Math Physics Test - due 11/13/2004

Fourier and Laplace Transforms
Take Home Part

October 31, 2007

1. A harmonic oscillator, initially at rest, is driven by the force $F_0\left(t/T\right)$, for $0 < t \leq T$, and then F_0, for $t > T$. Using the Laplace transform method, find the resulting oscillations (for $t > T$).

   Solution

   Equation of motion

   $$\ddot{x} + \omega^2 x = \frac{F\left(t\right)}{m}$$

   and its Laplace transform

   $$\begin{aligned} \left(s^2 + \omega^2\right) X &= \frac{F_0}{m}\left[\int_0^T \frac{t}{T}\exp\left(-st\right)dt + \int_T^\infty \exp\left(-st\right)dt\right] \\ &= \frac{F_0}{m}\frac{1-\exp\left(sT\right)}{s^2 T} \end{aligned}$$

   give

   $$X = \frac{F_0}{Tm}\frac{\left[1-\exp\left(-sT\right)\right]}{s^2\left(s^2+\omega^2\right)}$$

   For $t > T$, from Laplace inversion formula, find

   $$\begin{aligned} x &= \frac{F_0}{Tm}\oint_C \frac{\left[1-\exp\left(-sT\right)\right]}{s^2\left(s^2+\omega^2\right)}\exp\left(st\right)d \\ s &= \frac{F_0}{m\omega^2} + \frac{F_0}{mT\omega^3}\sin\left[\omega\left(t-T\right) - \sin\omega t\right] \\ &= \frac{F_0}{m\omega^2} - \frac{2F_0}{mT\omega^3}\sin\frac{\omega T}{2}\cos\omega\left(t-\frac{T}{2}\right) \end{aligned}$$