Integration

Math Physics
Quiz 2

10-30-2007

1. The concentration of gas in a cylinder of radius R and height h, rotating with angular velocity ω, is

$$n(r) = C \exp\left(\frac{m\omega^2 r^2}{2T}\right)$$

where r is the distance from the cylinder's axis, m is a molecules mass and T is temperature (in energy units). Given the total number N of molecules in the cylinder, evaluate the constant C and find $n(r)$. Express your answer in terms of

$$n_0 \equiv \frac{N}{V}$$

where V is the volume of the cylinder. Defining a dimensionless parameter

$$\alpha \equiv \frac{m\omega^2 R^2}{2T}$$

investigate your answer in the limit $\alpha \ll 1$ and $\alpha \gg 1$.

Solution

Normalization gives

$$\begin{aligned} N &= \int_V n(r)\, dV = 2\pi h C \int_0^R \exp\left(\frac{m\omega^2 r^2}{2T}\right) r dr = 2\pi h C \int_0^{R^2/2} \exp\left(\frac{m\omega^2 z}{T}\right) dz \\ &= \frac{2\pi h C T}{m\omega^2}\left[\exp\left(\frac{m\omega^2 R^2}{2T}\right) - 1\right] \end{aligned}$$

and

$$n(r) = \frac{N m\omega^2}{2\pi h T} \frac{\exp\left(m\omega^2 r^2/2T\right)}{\exp\left(m\omega^2 R^2/2T\right) - 1} = n_0 \frac{\alpha}{\exp\alpha - 1} \exp\left(\frac{\alpha r^2}{R^2}\right)$$

with the limits

$$\begin{aligned} n(r) &\xrightarrow{\alpha \ll 1} n_0 \\ n(r) &\xrightarrow{\alpha \gg 1} n_0 \alpha \exp\left[-\alpha\left(1 - \frac{r^2}{R^2}\right)\right] \end{aligned}$$

2. The partition function of an ideal monatomic gas, confined to a volume V at temperature T, is given by

$$Z = \frac{V}{(2\pi\hbar)^3} \int d^3p \exp\left(-\frac{p^2}{2mT}\right)$$

where m is the mass of each atom. Evaluate the Helmholtz free energy of the gas, $F = -NT \log Z$, where N is the total number of atoms in the gas, and find its equation of state using

$$P = -\left(\frac{\partial F}{\partial V}\right)_T$$

Solution

$$\begin{aligned}
Z &= \frac{4\pi V}{(2\pi\hbar)^3} \int_0^\infty dp p^2 \exp\left(-\frac{p^2}{2mT}\right) \\
&= \frac{(2mT)^{3/2} V}{2\pi^2\hbar^3} \int_0^\infty dp p^2 \exp\left(-p^2\right) \\
&= -\frac{(2mT)^{3/2} V}{2\pi^2\hbar^3} \frac{\partial}{\partial\alpha} \int_0^\infty dp \exp\left(-\alpha p^2\right) |_{\alpha=1} \\
&= -\frac{(2mT)^{3/2} V}{2\pi^2\hbar^3} \frac{\partial}{\partial\alpha} \left(\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right) |_{\alpha=1} \\
&= \frac{(2mT)^{3/2} V\sqrt{\pi}}{8\pi^2\hbar^3} = V\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}
\end{aligned}$$

whereof

$$\begin{aligned}
F &= -NT \log V\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} \\
P &= \frac{NT}{V}
\end{aligned}$$

3. To find the Green's function of the Helmholtz equation in 3D, the following integral must be evaluated:

$$G_3(r) = -\frac{1}{(2\pi)^2 r} \int_{-\infty}^{\infty} dq \frac{q \sin qr}{q^2 - k^2}$$

Evaluate $G_3(r)$ for the cases below (pick any two):

(a) when both poles are moved slightly off the axis to the upper plane;

(b) when the poles are moved slightly off the axis; the pole at k to the upper plane, and the pole $-k$ to the lower plane;

(c) when the poles are moved slightly off the axis; the pole at k to the lower plane, and the pole $-k$ to the upper plane;

(d) when the integral is understood as the principal value.

Solution

(a)

$$G_3(r) = -\frac{1}{4\pi r}\left\{\text{res}\left[\frac{q\left[\exp(iqr)\right]}{q^2-k^2},k\right] + \text{res}\left[\frac{q\left[\exp(iqr)\right]}{q^2-k^2},-k\right]\right\} = -\frac{\cos kr}{4\pi r}$$

(b)

$$\begin{aligned} G_3(r) &= -\frac{1}{4\pi r}\left\{\text{res}\left[\frac{q\left[\exp(iqr)\right]}{q^2-k^2},k\right] - \text{res}\left[\frac{q\left[-\exp(-iqr)\right]}{q^2-k^2},-k\right]\right\} \\ &= -\frac{1}{4\pi r}\left[\frac{\exp(ikr)}{2}+\frac{\exp(ikr)}{2}\right] = -\frac{\exp(ikr)}{4\pi r} \end{aligned}$$

(c)

$$\begin{aligned} G_3(r) &= -\frac{1}{4\pi r}\left\{\text{res}\left[\frac{q\left[\exp(iqr)\right]}{q^2-k^2},-k\right] - \text{res}\left[\frac{q\left[-\exp(-iqr)\right]}{q^2-k^2},k\right]\right\} \\ &= -\frac{1}{4\pi r}\left[\frac{\exp(-ikr)}{2}+\frac{\exp(-ikr)}{2}\right] = -\frac{\exp(-ikr)}{4\pi r} \end{aligned}$$

(d)

$$\begin{aligned} G_3(r) &= \frac{i}{2(2\pi)^2 r}\left[P\int_C dq\frac{q\exp(iqr)}{q^2-k^2} - P\int_C dq\frac{q\exp(-iqr)}{q^2-k^2}\right] \\ &= \frac{i}{2(2\pi)^2 r}\left\{\pi i\left(\text{res}\left[\frac{q\exp(iqr)}{q^2-k^2},k\right] + \text{res}\left[\frac{q\exp(iqr)}{q^2-k^2},-k\right]\right)\right\} \\ &\quad -\frac{i}{2(2\pi)^2 r}\left\{-\pi i\left(\text{res}\left[\frac{q\exp(-iqr)}{q^2-k^2},k\right] + \text{res}\left[\frac{q\exp(-iqr)}{q^2-k^2},-k\right]\right)\right\} \\ &= -\frac{1}{8\pi r}\left(\frac{\exp(ikr)}{2}+\frac{\exp(-ikr)}{2}+\frac{\exp(-ikr)}{2}+\frac{\exp(ikr)}{2}\right) = -\frac{\cos kr}{4\pi r} \end{aligned}$$

4. Fresnel integrals, used in many fields of physics (e.g. optics), are defined as follows:

$$\begin{aligned} \mathrm{S}(x) &= \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \\ \mathrm{C}(x) &= \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \end{aligned}$$

Find their approximate expansions for small and large arguments. (Limit your expansions to $O\left(x^3\right)$ for $x \ll 1$ and $O\left(x^{-1}\right)$ for $x \gg 1$). *Hint*: to evaluate $\mathrm{S}\left(\infty\right)$ and $\mathrm{C}\left(\infty\right)$, evaluate the integral $\oint \exp\left(-z^2/2\right) dz$ along the contour consisting of (i) the real axis from 0 to $+\infty$, (ii) one quarter of a large circle at $|z| = \infty$, and (iii) return to the origin along the line $\arg\left(z\right) = \pi/4$.

Solution

For $x \ll 1$,

$$\begin{aligned} \mathrm{S}\left(x\right) &= \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \approx \int_0^x \frac{\pi t^2}{2} dt = \frac{\pi x^3}{6} \\ \mathrm{C}\left(x\right) &= \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \approx \int_0^x dt = x \end{aligned}$$

To evaluate $\mathrm{S}\left(\infty\right)$ and $\mathrm{C}\left(\infty\right)$,

$$\begin{aligned} 0 &= \oint \exp\left(-\frac{z^2}{2}\right) dz = \int_0^\infty \exp\left(-\frac{t^2}{2}\right) dt - \frac{1}{\sqrt{2}} \int_0^\infty \left[\cos\left(\frac{t^2}{2}\right) + \sin\left(\frac{t^2}{2}\right)\right] dt \\ &+ \frac{i}{\sqrt{2}} \int_0^\infty \left[\sin\left(\frac{t^2}{2}\right) - \cos\left(\frac{t^2}{2}\right)\right] dt \end{aligned}$$

and equating Re and Im parts to zero in the r.h.s.,

$$\int_0^\infty \sin\left(\frac{\pi t^2}{2}\right) dt = \int_0^\infty \cos\left(\frac{\pi t^2}{2}\right) dt = \frac{1}{2}$$

For $x \gg 1$,

$$\begin{aligned} \mathrm{S}\left(\infty\right) - \mathrm{S}\left(x\right) &= \int_x^\infty \sin\left(\frac{\pi t^2}{2}\right) dt = -\frac{1}{\pi} \int_x^\infty \frac{1}{t} d\cos\left(\frac{\pi t^2}{2}\right) \approx \frac{\cos \pi x^2/2}{\pi x} \\ \mathrm{S}\left(x\right) &\approx \frac{1}{2} - \frac{\cos \pi x^2/2}{\pi x} \end{aligned}$$

$$\begin{aligned} \mathrm{C}\left(\infty\right) - \mathrm{C}\left(x\right) &= \int_x^\infty \cos\left(\frac{\pi t^2}{2}\right) dt = \frac{1}{\pi} \int_x^\infty \frac{1}{t} d\sin\left(\frac{\pi t^2}{2}\right) \approx \frac{\sin \pi x^2/2}{\pi x} \\ \mathrm{C}\left(x\right) &\approx \frac{1}{2} + \frac{\sin \pi x^2/2}{\pi x} \end{aligned}$$