Ordinary Differential Equations

Math Physics

Quiz 1

(Dated: 10-09-2007)

1. A thin, homogenous cable of mass m is at rest on a frictionless table surface, perpendicular to the table's edge. Initially, one end of the cable is right at the edge. The cable gets a slight push, just over the edge, giving it a very small speed v. Given that the cable's length l is much smaller than the height of the table and that the cable cannot be stretched or compressed and slides in a straight line, find:

   (a) Acceleration of the cable at the moment when its other end is falls off the table.

   (b) Velocity of the cable at that moment.

   Bonus: Give the energy interpretation of your results. Investigate your answer in the limit of very small v (define the limit).

   Solution

   Equation of motion

$$m\ddot{x} = \frac{mgx}{l}$$

   where x is the cable's length that has already slid off the table. Denote

$$\tau = \sqrt{\frac{l}{g}} \tag{1}$$

   then

$$x = A\exp\left(\frac{t}{\tau}\right) + B\exp\left(-\frac{t}{\tau}\right)$$

   Using initial conditions

$$x\left(0\right) = 0,\ \dot{x}\left(0\right) = v$$

   find

$$x = v\tau\sinh\left(\frac{t}{\tau}\right)$$

   The other end is off of the table when $x = l$ at the time t_f

$$l = v\tau\sinh\left(\frac{t_f}{\tau}\right) \tag{2}$$

Acceleration at that moment is, obviously, g. Verify

$$\dot{x} = v \cosh\left(\frac{t}{\tau}\right)$$
$$\ddot{x} = \frac{v}{\tau} \sinh\left(\frac{t}{\tau}\right)$$

Using (1) and (2),

$$\ddot{x}(t_f) = \frac{v}{\tau} \sinh\left(\frac{t_f}{\tau}\right) = \frac{v}{\tau} \frac{l}{v\tau} = \frac{l}{\tau^2} = g$$

Also

$$\dot{x}(t_f) = v \cosh\left(\frac{t_f}{\tau}\right) = v\sqrt{1 + \sinh^2\left(\frac{t_f}{\tau}\right)} = v\sqrt{1 + \left(\frac{l}{v\tau}\right)^2} = \sqrt{v^2 + gl}$$

Notice that

$$\frac{m\dot{x}^2(t_f)}{2} = \frac{mv^2}{2} + \frac{mgl}{2}$$

is just the energy conservation law, as the cable's c.m. is $l/2$ from the table's edge at the time t_f. For small v, $v \ll \sqrt{gl}$

$$\dot{x}(t_f) \simeq \sqrt{gl}$$

2. A harmonic oscillator

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

initially at rest ($x = \dot{x} = 0$ at $t = 0$), is driven by a constant force, which acts for a finite time T:

$$F(t) = \begin{matrix} F_0\,,\ 0 < t < T \\ 0,\ t < 0,\ t > T \end{matrix}$$

Determine the final amplitude of the oscillations, $t > T$.

Solution

For $0 < t < T$, the solution of the inhomogeneous equation, satisfying initial conditions, is

$$x(t) = \frac{F_0}{m\omega^2}\left[1 - \cos(\omega t)\right]$$

For $t > T$, the solution is

$$x(t) = A \sin\left[\omega(t - T)\right] + B \cos\left[\omega(t - T)\right]$$

an is the subject to the conditions

$$x(T) = \frac{F_0}{m\omega^2}\left[1 - \cos(\omega T)\right],\ \dot{x}(T) = \frac{F_0}{m\omega}\sin(\omega T)$$

which gives

$$x(t) = \frac{F_0}{m\omega^2}\left\{\left[1 - \cos(\omega T)\right]\cos\left[\omega(t - T)\right] + \sin(\omega T)\sin\left[\omega(t - T)\right]\right\}$$

The amplitude is

$$A = \frac{F_0}{m\omega^2}\sqrt{\left[1 - \cos(\omega T)\right]^2 + \sin^2(\omega T)} = \frac{F_0}{m\omega^2}\sqrt{2\left[1 - \cos(\omega T)\right]} = \frac{2F_0}{m\omega^2}\sin\left(\frac{\omega T}{2}\right)$$

3. A particle of mass m, moving in one dimension, is subject to a force $F(x)$. Show that integration of the equation of motion gives the energy conservation law. *Hint*: use the method of quadrature.

   Solution

   Multiplying both sides of

$$m\ddot{x} = F(x)$$

   by $\dot{x}$ yields

$$\begin{aligned}\frac{1}{2}m\frac{d\left(\dot{x}^2\right)}{dt} &= F(x)\frac{dx}{dt} \\ \frac{1}{2}md\left(\dot{x}^2\right) &= F(x)\,dx\end{aligned}$$

   Integration gives the energy conservation law

$$\frac{1}{2}m\dot{x}^2 = \int F(x)\,dx = -U(x) + E$$