An eigen-value problem

$$Lu \equiv -(pu')' + qu = \lambda u, \qquad 0 < x < l \qquad (1)$$

$$\left.\begin{aligned} h_1 u(0) - h_2 u'(0) = 0 \\ H_1 u(l) + H_2 u'(l) = 0 \end{aligned}\right\} \qquad (2)$$

is called the Sturm-Liouville problem

Here $p(x) > 0$, $q(x) \geq 0$

$h_{1,2}$, $H_{1,2} \geq 0$

$h_1 + h_2$, $H_1 + H_2 > 0$

<u>Green's</u> <u>function</u>

Consider a b.v. problem,

$$Lu = f(x) \qquad (3)$$

Let v_1, v_2 be solutions of

$$Lv = 0$$

$$\left.\begin{aligned} h_1 v_1(0) - h_2 v_1'(0) = 0 \\ H_1 v_2(l) + H_2 v_2'(l) = 0 \end{aligned}\right\} \qquad (4)$$

Assume that $\lambda = 0$ is not an e.v., then v_1 and v_2 are linearily independent and

$$w(x) = \begin{vmatrix} v_1(x) & v_2(x) \\ v_1'(x) & v_2'(x) \end{vmatrix} \neq 0 \; *$$

1'

* Wronskian

Suppose k_i exist that

$$\sum_{i=1}^{n} k_i v_i = 0$$

Differentiate n times

$$\sum_{i=1}^{n} k_i v_i = 0$$

$$\sum_{i=1}^{n} k_i v_i' = 0 \quad \Rightarrow W = \begin{vmatrix} v_1, \dots, v_n \\ v_1', \dots, v_n' \\ \vdots \\ v_1^{(n-1)}, \dots, v_n^{(n-1)} \end{vmatrix} = 0$$

$$\vdots$$

$$\sum_{i=1}^{n} k_i v_i^{(n-1)} = 0$$

The combination $p(x)w(x)$ is a constant

$$\frac{d}{dx}\left[p(x)w(x)\right] = p'w + pw'$$

$$w' = (v_1 v_2' - v_1' v_2)' = v_1 v_2'' - v_1'' v_2$$

$$-p'v' - pv'' + qv = 0, \quad v'' = -\frac{p'}{p}v' + \frac{q}{p}v$$

$$w' = v_1\left(-\frac{p'}{p}v_2' + \frac{q}{p}v_2\right) - v_2\left(-\frac{p'}{p}v_1' + \frac{q}{p}v_1\right)$$

$$\frac{d}{dx}\left[p(x)w(x)\right] = p'(v_1v_2' - v_1'v_2)$$

$$+ v_1(-p'v_2' + qv_2) - v_2(-p'v_1' + qv_1) = 0$$

Use variational technique

$$u(x) = C_1(x)v_1(x) + C_2(x)v_2(x)$$

$$u'(x) = \underbrace{C_1'(x)v_1(x) + C_2'(x)v_2(x)}_{=0} + C_1(x)v_1'(x) + C_2(x)v_2'(x)$$

$$u''(x) = C_1'v_1' + C_2'v_2' + C_1v_1'' + C_2v_2''$$

$$-pu'' - p'u' + qu = f$$

$$C_1'v_1' + C_2'v_2' + C_1v_1'' + C_2v_2'' - \frac{p'}{p}C_1v_1' - \frac{p'}{p}C_2v_2' + \frac{q}{p}C_1v_1 + \frac{q}{p}C_2v_2 = \frac{f}{p}$$

Recall that $v'' - \frac{p'}{p} v' + \frac{q}{p} = 0$

Then,

$$\left.\begin{aligned} C_1' v_1' + C_2' v_2' &= -\frac{f}{p} \\ C_1' v_1 + C_2' v_2 &= 0 \end{aligned}\right\} \text{ w/r to } C_1', C_2'$$

$$C_{1,2}' = \pm \frac{f(x)\, v_{2,1}(x)}{p(x)\, w(x)} = \pm \frac{f(x)\, v_{2,1}(x)}{p(0)\, w(0)}$$

B.v.

$$h_1 u(0) - h_2 u'(0) = \Big\{ h_1 \left[c_1 v_1 + c_2 v_2 \right] - h_2 \left[c_1 v_1' + c_1' v_1 + c_2 v_2' + c_2' v_2 \right] \Big\}\Big|_0$$

Since $C_1' v_1 + c_2' v_2 = 0$,

$$h_1 u(0) - h_2 u'(0) = \underbrace{C_1 \left[h_1 v_1 - h_2 v_1' \right]\Big|_0}_{0} + C_2 \left[h_1 v_2 - h_2 v_2' \right]\Big|_0$$

$$= \Big\{ C_2 \left[h_1 v_2 - h_2 v_2' \right] \Big\}\Big|_0$$

Here, $C_2(0) = 0$. Also, by analogy, $C_1(l) = 0$

$$C_1(x) = -\frac{1}{p(0)\, w(0)} \int_x^l f(y)\, v_2(y)\, dy$$

$$C_2(x) = -\frac{1}{p(0)\, w(0)} \int_0 f(y)\, v_1(y)\, dy$$

$$u(x) = \int_0^l G(x,y) f(y)\,dy$$

$$G(x,y) = -\frac{1}{p(0)W(0)} \begin{cases} v_1(x)v_2(y), & 0 \le x \le y \\ v_2(x)\,v_1(y), & y \le x \le l \end{cases}$$

SL problem is equiv. to e.v. problem for the integral equation

$$u(x) = \lambda \int_0^l G(x,y)\,u(y)\,dy$$