Assignment 4

Special Functions

1. Prove the addition theorems:

$$J_n(u+v) = \sum_{k=-\infty}^{\infty} J_k(u) J_{n-k}(v)$$

$$J_0(u+v) = J_0(u) J_0(v) + \sum_{k=1}^{\infty} J_k(u) J_{-k}(v)$$

Hint: Use the generating function for the Bessel functions.

2. Derive the following relationships:

$$\exp(ix\cos\theta) = \sum_{k=-\infty}^{\infty} i^k J_k(x) \exp(ik\theta)$$

$$\cos x = J_0(x) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(x)$$

$$\sin x = 2\sum_{k=1}^{\infty} (-1)^{k+1} J_{2k+1}(x)$$

Hint: Use the generating function for the Bessel functions.

3. Prove that

$$\frac{\sin x}{x} = \int_0^{\pi/2} J_0(x\cos\theta)\cos\theta d\theta$$

$$\frac{1-\cos x}{x} = \int_0^{\pi/2} J_1(x\cos\theta)\, d\theta$$

Hint: Use

$$\int_0^{\pi/2} \cos^{2k+1}\theta d\theta = \frac{(2k)!!}{(2k+1)!!}$$

4. A function $f(x)$ is expanded in a Bessel series:

$$f(x) = \sum_{k=1}^{\infty} a_k J_m(\alpha_{mk} x)$$

where α_{mk} is the kth root of J_m. Prove the Parseval relation

$$\int_0^1 f^2(x)\, x dx = \frac{1}{2} \sum_{k=1}^{\infty} a_k^2 J_{m+1}^2(\alpha_{mk})$$

5. Prove that

$$\sum_{k=1}^{\infty} \alpha_{mk}^{-2} = \frac{1}{4(m+1)}$$

Hint: Expand x^m in a Bessel series and use the Parseval relation.

6. Show that

$$\int_{-\infty}^{\infty} j_n^2(x)\, dx = \frac{\pi}{2n+1}$$

7. Prove that

$$P_l'(1) = \frac{1}{2} l(l+1)$$

8. Verify the following expansions

$$\delta(1-x) = \sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x)$$

$$\delta(1+x) = \sum_{l=0}^{\infty} (-1)^l \frac{2l+1}{2} P_l(x)$$

9. A function $f(x)$ is expanded in a Legendre series:

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x)$$

Prove the Parseval relation

$$\int_{-1}^{1} f^2(x)\, dx = \sum_{l=0}^{\infty} \frac{2}{2l+1} a_l^2$$

10. Neutrons (mass 1) are being scattered by a nucleus of mass A ($A > 1$). In the c.m. system the scattering is isotropic. Then, in the lab system the average of the cosine of the angle of deflection of the neutron is

$$\langle \cos\psi \rangle = \frac{1}{2}\int_0^\pi \frac{A\cos\theta + 1}{\left(A^2 + 2A\cos\theta + 1\right)^{1/2}} \sin\theta d\theta$$

Show that

$$\langle \cos\psi \rangle = \frac{2}{3A}$$

Hint: Expand the denominator.