Fourier and Laplace Transforms

Math Physics
Quiz 3

11-14-2006

1. Two particles, B and C, of equal mass m are connected by a spring BC of strength k_1. They are also connected to opposite walls at, respectively, points A and D by springs AB and CD, each of strength k_2. The particles move along a straight line in a horizontal, frictionless groove. Using the Laplace transform method, find the motion of the system in the following cases:

   (a) At $t = 0$ one of the particles moves with velocity v, while the other particle is at rest and the initial displacement from equilibrium of either particle is zero.

   (b) At $t = 0$ one of the particles is displaced from its equilibrium position over a distance a, while the initial displacement from equilibrium of the other particle is zero and both particles are initially at rest.

   Bonus: Solve without the use of Laplace transform and confirm that you get the same result.

   Solution

   Lagrangian

$$\mathcal{L} = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{k_1 \left(x_1 - x_2\right)^2}{2} - \frac{k_2 x_1^2}{2} - \frac{k_2 x_2^2}{2}$$

   Equations of motion

$$\begin{aligned} \ddot{x}_1 + \frac{k_1}{m}\left(x_1 - x_2\right) + \frac{k_2}{m} x_1 &= 0 \\ \ddot{x}_2 + \frac{k_1}{m}\left(x_2 - x_1\right) + \frac{k_2}{m} x_2 &= 0 \end{aligned}$$

   (a) Laplace transform

$$\begin{aligned} s^2 X_1 - v + \frac{k_1}{m}\left(X_1 - X_2\right) + \frac{k_2}{m} X_1 &= 0 \\ s^2 X_2 + \frac{k_1}{m}\left(X_2 - X_1\right) + \frac{k_2}{m} X_2 &= 0 \end{aligned}$$

whereof

$$\begin{aligned} X_1 &= \frac{v\left[s^2+(k_1+k_2)/m\right]}{\left[s^2+(2k_1+k_2)/m\right]\left[s^2+k_2/m\right]} \\ &= \frac{v}{2}\left[\frac{1}{s^2+k_2/m}+\frac{1}{s^2+(2k_1+k_2)/m}\right] \\ X_2 &= \frac{vk_1/m}{\left[s^2+(2k_1+k_2)/m\right]\left[s^2+k_2/m\right]} \\ &= \frac{v}{2}\left[\frac{1}{s^2+k_2/m}-\frac{1}{s^2+(2k_1+k_2)/m}\right] \end{aligned}$$

Laplace inversion formula

$$x_{1,2}=\frac{v}{2}\left(\frac{\sin\omega_1 t}{\omega_1}\pm\frac{\sin\omega_2 t}{\omega_2}\right)$$

where $\omega_{1,2}$ are frequencies of normal modes

$$\omega_1=\sqrt{\frac{k_2}{m}},\ \omega_2=\sqrt{\frac{k_2+2k_1}{m}}$$

(b) Laplace transform

$$\begin{aligned} s^2X_1-sa+\frac{k_1}{m}(X_1-X_2)+\frac{k_2}{m}X_1 &= 0 \\ s^2X_2+\frac{k_1}{m}(X_2-X_1)+\frac{k_2}{m}X_2 &= 0 \end{aligned}$$

whereof

$$\begin{aligned} X_1 &= \frac{as\left[s^2+(k_1+k_2)/m\right]}{\left[s^2+(2k_1+k_2)/m\right]\left[s^2+k_2/m\right]} \\ &= \frac{a}{2}\left[\frac{s}{s^2+k_2/m}+\frac{s}{s^2+(2k_1+k_2)/m}\right] \\ X_2 &= \frac{ask_1/m}{\left[s^2+(2k_1+k_2)/m\right]\left[s^2+k_2/m\right]} \\ &= \frac{a}{2}\left[\frac{s}{s^2+k_2/m}-\frac{s}{s^2+(2k_1+k_2)/m}\right] \end{aligned}$$

Laplace inversion formula

$$x_{1,2}=\frac{a}{2}\left(\cos\omega_2 t\pm\cos\omega_1 t\right)$$

2. Evaluate the expectation value of kinetic energy for a normalized 3D wave packet

$$\psi(\mathbf{r}) = \left(\frac{2}{\pi a^2}\right)^{3/4} \exp\left(-\frac{r^2}{a^2}\right)$$

using two methods:

(a) Evaluate

$$K = \int \psi^*(\mathbf{r}) \frac{\widehat{\mathbf{p}}^2}{2m} \psi(\mathbf{r})\, d^3\mathbf{r} = \frac{\hbar^2}{2m} \int |\boldsymbol{\nabla}\psi(\mathbf{r})|^2\, d^3\mathbf{r}$$

where $\widehat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$.

(b) Using $\psi(\mathbf{k}) = \mathcal{F}[\psi(\mathbf{r})]$, evaluate

$$K = \frac{1}{(2\pi)^3} \int \psi^*(\mathbf{k}) \frac{\hbar^2 k^2}{2m} \psi(\mathbf{k})\, d^3\mathbf{k}$$

(Remember that by Parseval's theorem $(2\pi)^{-3} \int |\psi(\mathbf{k})|\, d^3\mathbf{k} = \int |\psi(\mathbf{r})|^2\, d^3\mathbf{r}$.)

Solution

(a) $\boldsymbol{\nabla}\psi(\mathbf{r}) = -\left(2\mathbf{r}/a^2\right)\left(2/\pi a^2\right)^{3/4} \exp\left(-r^2/a^2\right)$

$$\begin{aligned} K &= \frac{2\hbar^2}{ma^4}\left(\frac{2}{\pi a^2}\right)^{3/2} \int r^2 \exp\left(-\frac{2r^2}{a^2}\right) d^3\mathbf{r} \\ &= \frac{8\pi\hbar^2}{ma^4}\left(\frac{2}{\pi a^2}\right)^{3/2} \int_0^\infty r^4 \exp\left(-\frac{2r^2}{a^2}\right) dr \\ &= \frac{8\pi\hbar^2}{ma^4}\left(\frac{2}{\pi a^2}\right)^{3/2} \sqrt{\frac{\pi}{2}}\frac{3a^5}{32} = \frac{3\hbar^2}{2ma^2} \end{aligned}$$

(b) Per pp. 106-107 in M&W, evaluation of $\mathcal{F}[\psi(\mathbf{r})]$ gives

$$\psi(\mathbf{k}) = \left(2\pi a^2\right)^{3/4} \exp\left(-\frac{k^2 a^2}{4}\right)$$

whereof

$$\begin{aligned} K &= \frac{\hbar^2}{2m(2\pi)^3}\left(2\pi a^2\right)^{3/2} \int k^2 \exp\left(-\frac{k^2 a^2}{2}\right) d^3\mathbf{k} \\ &= \frac{\hbar^2}{m(2\pi)^2}\left(2\pi a^2\right)^{3/2} \int_0^\infty k^4 \exp\left(-\frac{k^2 a^2}{2}\right) dk \\ &= \frac{\hbar^2}{m(2\pi)^2}\left(2\pi a^2\right)^{3/2} \sqrt{\frac{\pi}{2}}\frac{3}{a^5} = \frac{3\hbar^2}{2ma^2} \end{aligned}$$