Integration

Math Physics
Quiz 2

11-02-2006

1. Evaluate the integral

$$I = \int_0^\infty dx \frac{\sqrt{x}}{(1+x)^2}$$

Solution

Using contour in Fig. 3-5 on p. 71,

$$\begin{aligned} 2I &= \oint dz \frac{\sqrt{z}}{(1+z)^2} = 2\pi i \operatorname{res}\left[\frac{\sqrt{z}}{(1+z)^2},\, z=-1\right] = \pi i \frac{1}{\sqrt{\exp i\pi}} \\ &= \pi i \exp\left(-i\frac{\pi}{2}\right) = \pi i\,(-i) = \pi \end{aligned}$$

Verify by substituting $x = y^2$,

$$\begin{aligned} I &= 2\int_0^\infty dy \frac{y^2}{(1+y^2)^2} = -2\frac{\partial}{\partial a}\int_0^\infty \frac{dy}{(1+ay^2)}\,|_{a=1} \\ &= -2\frac{\partial}{\partial a}\left(a^{-1/2}\int_0^\infty \frac{dy}{1+y^2}\right)|_{a=1} = \int_0^\infty \frac{dy}{1+y^2} \overset{y=\tan\theta}{=} \int_0^{\pi/2} d\theta = \frac{\pi}{2} \end{aligned}$$

2. Evaluate the integral

$$I(a,b) = \int_0^\infty dx \frac{\exp\left(-ax^2\right) - \exp\left(-bx^2\right)}{x^2};\ a,b>0$$

Solution

$$\frac{\partial I(a,b)}{\partial a} = -\int_0^\infty dx \exp\left(-ax^2\right) = -\frac{1}{2}\sqrt{\frac{\pi}{a}},\ I(a,b) = -\sqrt{\pi a} + f(b)$$

$$\frac{\partial I(a,b)}{\partial b} = \frac{df(b)}{db} = \int_0^\infty dx \exp\left(-bx^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{b}},\ f(b) = \sqrt{\pi b} + C$$

$$I(a,b) = \sqrt{\pi}\left(\sqrt{b}-\sqrt{a}\right) + C = \sqrt{\pi}\left(\sqrt{b}-\sqrt{a}\right),\ \text{since } I(a,a) = 0$$

3. Evaluate the integral
$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2}$$
by treating poles on the axis of integration

   (a) As principal values;
   (b) By moving pole at $-\sigma$ down and pole at σ up;
   (c) By moving pole at $-\sigma$ up and pole at σ down.

Solution
$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2} = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{x\left[\exp(ix) - \exp(-ix)\right]}{x^2 - \sigma^2}$$

   (a) Evaluating as principal values:

Closing contour in the upper plane (counterclockwise)
$$\begin{aligned} & \frac{1}{2i} \int_{-\infty}^{\infty} \frac{z \exp(iz)}{z^2 - \sigma^2} \\ = & \frac{(\pi i)}{2i} \left\{ \text{res}\left[\frac{z \exp(iz)}{z^2 - \sigma^2}, \ z = -\sigma\right] + \text{res}\left[\frac{z \exp(iz)}{z^2 - \sigma^2}, \ z = \sigma\right]\right\} \\ = & \frac{\pi}{2}\left[\frac{-\sigma \exp(-i\sigma)}{-2\sigma} + \frac{\sigma \exp(i\sigma)}{2\sigma}\right] = \frac{\pi}{2} \cos \sigma \end{aligned}$$

Closing contour in the lower plane plane (clockwise)
$$\begin{aligned} & -\frac{1}{2i} \int_{-\infty}^{\infty} \frac{z \exp(-iz)}{z^2 - \sigma^2} \\ = & -\frac{(-\pi i)}{2i} \left\{ \text{res}\left[\frac{z \exp(-iz)}{z^2 - \sigma^2}, \ z = -\sigma\right] + \text{res}\left[\frac{z \exp(-iz)}{z^2 - \sigma^2}, \ z = \sigma\right]\right\} \\ = & \frac{\pi}{2}\left[\frac{-\sigma \exp(i\sigma)}{-2\sigma} + \frac{\sigma \exp(-i\sigma)}{2\sigma}\right] = \frac{\pi}{2} \cos \sigma \end{aligned}$$

Adding the two, $I = \pi \cos \sigma$.

   (a) Evaluating by moving pole at $-\sigma$ down and pole at σ up:
$$\begin{aligned} I & = \frac{(2\pi i)}{2i} \text{res}\left[\frac{z \exp(iz)}{z^2 - \sigma^2}, \ z = \sigma\right] - \frac{(-2\pi i)}{2i} \text{res}\left[\frac{z \exp(-iz)}{z^2 - \sigma^2}, \ z = -\sigma\right] \\ & = \pi \exp i\sigma \end{aligned}$$

   (b) Evaluating by moving pole at $-\sigma$ up and pole at σ down:
$$\begin{aligned} I & = \frac{(2\pi i)}{2i} \text{res}\left[\frac{z \exp(iz)}{z^2 - \sigma^2}, \ z = -\sigma\right] - \frac{(-2\pi i)}{2i} \text{res}\left[\frac{z \exp(-iz)}{z^2 - \sigma^2}, \ z = \sigma\right] \\ & = \pi \exp(-i\sigma) \end{aligned}$$

4. Find the expansions for the integral

$$I(x) = \int_0^x dy \frac{\sin^2 y}{y^2}$$

useful for large and small x. Limit you expansion to three terms for the former case and two terms for the latter case.

Solution

For $x \ll 1$,

$$I(x) \approx \int_0^x dy \frac{y^2 - y^4/3}{y^2} = x - \frac{x^3}{9}$$

For $x \gg 1$,

$$I(x) = \int_0^\infty dy \frac{\sin^2 y}{y^2} - \int_x^\infty dy \frac{\sin^2 y}{y^2} = \frac{1}{2}\int_{-\infty}^\infty dy \frac{\sin^2 y}{y^2} - \int_x^\infty dy \frac{\sin^2 y}{y^2}$$

First, with C "walking around" 0 in the lower plane,

$$\begin{aligned}
\frac{1}{2}\int_{-\infty}^\infty dy \frac{\sin^2 y}{y^2} &= \frac{1}{2}\int_C dz \frac{\sin^2 z}{z^2} = -\frac{1}{8}\int_C dz \frac{[\exp iz - \exp(-iz)]^2}{z^2} \\
&= -\frac{1}{8}\oint_U dz \frac{\exp 2iz}{z^2} - \frac{1}{8}\oint_L dz \frac{\exp(-2iz)}{z^2} + \frac{1}{4}\oint \frac{dz}{z^2} \\
&= -\frac{2\pi i}{8} \operatorname{res}\left[\frac{\exp 2iz}{z^2},\ z = 0\right] = -\frac{2\pi i}{8}(2i) = \frac{\pi}{2}
\end{aligned}$$

Then,

$$\begin{aligned}
I(x) &= \frac{\pi}{2} - \int_x^\infty dy \frac{\sin^2 y}{y^2} = \frac{\pi}{2} - \int_x^\infty \frac{dy}{2y^2} + \int_x^\infty dy \frac{\cos 2y}{2y^2} \\
&= \frac{\pi}{2} - \frac{1}{2x} + \int_x^\infty \frac{d(\sin 2y)}{4y^2} \\
&= \frac{\pi}{2} - \frac{1}{2x} + \frac{\sin 2y}{4y^2}\Big|_x^\infty - \int_x^\infty \frac{\sin 2y}{2y^3} \approx \frac{\pi}{2} - \frac{1}{2x} - \frac{\sin 2x}{4x^2}
\end{aligned}$$