Ordinary Differential Equations

Math Physics

Quiz 1

(Dated: 10-10-2006)

1. Solve the differential equation

$$y'' - \frac{4}{x}y' + 6\frac{y}{x^2} = \frac{\ln x}{x^2}$$

Solution

Rewrite eq. as

$$x^2 y'' - 4xy' + 6y = \ln x$$

Homogeneous eq.

$$x^2 y'' - 4xy' + 6y = 0$$

$y \propto x^m$

$$m\left(m-1\right) - 4m + 6 = 0$$

$$m_1 = 2,\ m_2 = 3$$

Variational approach

$$y = u_1 x^2 + u_2 x^3$$

gives

$$u_1' x^2 + u_2' x^3 = 0$$

$$2u_1' x + 3u_2' x^2 = \frac{\ln x}{x^2}$$

and

$$u_1' = -\frac{\ln x}{x^3},\ u_2' = \frac{\ln x}{x^4}$$

$$u_1 = \frac{\ln x}{2x^2} + \frac{1}{4x^2} + c_1,\ u_2 = -\frac{\ln x}{3x^3} - \frac{1}{9x^3} + c_2$$

$$y = \frac{1}{6}\ln x + \frac{5}{36} + c_1 x^2 + c_2 x^3$$

Alternative method: introduce $z = \ln x$. Then

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz},\ \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

and

$$y'' - 5y' + 6y = z$$

$$y = \frac{z}{6} + \frac{5}{36} + c_1 \exp\left(2z\right) + c_2 \exp\left(3z\right)$$

as before.

2. A harmonic oscillator

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

initially at rest ($x = \dot{x} = 0$ at $t = 0$), is driven by the force

$$F(t) = F_0 \exp(-\gamma t)$$

Find the oscillations at long times, $t \gg \gamma^{-1}$, and investigate your answer for $\omega \gg \gamma$ and $\omega \ll \gamma$.

Solution

Solution of the inhomogeneous equation satisfying initial conditions is

$$x = \frac{F_0}{m(\omega^2+\gamma^2)}\left[\exp(-\gamma t) - \cos\omega t + \frac{\gamma}{\omega}\sin\omega t\right] \overset{t\gg\gamma^{-1}}{\to} \frac{F_0}{m(\omega^2+\gamma^2)}\left[-\cos\omega t + \frac{\gamma}{\omega}\sin\omega t\right]$$

$$= \frac{F_0}{m\omega\sqrt{\omega^2+\gamma^2}}\sin\left(\omega t - \tan^{-1}\frac{\omega}{\gamma}\right) \approx \begin{matrix} -\left(F_0/m\omega^2\right)\cos\omega t, & \omega \gg \gamma \\ \left(F_0/m\omega\gamma\right)\sin\omega t, & \omega \ll \gamma \end{matrix}$$

3. For what values of the constant E does the differential equation

$$xy'' - (x - E)\, y = 0,\ x > 0$$

has a non-trivial solution vanishing at infinity? Find a solution such that $y(0) = 0$ and $y'(0) = 1$ with the smallest possible E.

Solution

Asymptotic behavior $y''_\infty - y_\infty = 0$, $y_\infty \propto \exp(-x)$. Look for a solution $y = u\exp(-x)$,

$$(-2u' + u'')\, x + Eu = 0$$

Look for a series solution

$$u = \sum_{m=0}^{\infty} c_m x^m$$

The recursion relationship

$$c_{m+1} m(m+1) = (-E + 2m)\, c_m$$

and the series diverges unless as $\exp(2x)$ unless $E = 2n$, n integer. For $y(0) = 0$, $y'(0) = 1$, the smallest possible value is $E = 2$, giving

$$y = x\exp(-x)$$