Assignment 5

Partial Differential Equations

1. Consider a string of length l stretched along the x-axis and subjected to a transverse displacement $u(x,t)$. The string is clamped at one end and attached at the other end to a massless ring which slides freely on a fixed rod. $u(x,t)$ satisfies the differential equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

where v is the speed of the wave.

(a) Write down the boundary conditions for $u(x,t)$.

(b) Find the general solution for the differential equation satisfying the given b.c.

(c) Assuming that at time $t = 0$

$$u(x,0) = \begin{matrix} \alpha x & 0 \leq x \leq l/2 \\ l\alpha/2 & l/2 \leq x \leq l \end{matrix}$$

$$\frac{\partial}{\partial t} u(x,0) = 0$$

find the solution for $u(x,t)$.

2. Given the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) u(x) = \phi(x)$$

subject to b.c.

$$u(l) = u(-l) = 0$$

$$-l \leq x \leq l$$

derive the Green's function and its spectral expansion (k^2 is different from any of the eigenvalues) and write the solution of the inhomogeneous differential equation.

3. (a) Find the electrostatic potential $V(\mathbf{r})$ inside a region, bounded by conducting plates $y = 0$, $y = d$ and $x = 0$ if the latter plate is maintained at the potential V_0 and the other two plates are grounded.(There are no charges inside the region.)

   (b) Find the potential assuming now that the boundary $y = d$ is maintained at the potential V_1 and consider the limiting case of $d \to \infty$.

   Hint: Look for a solution in the form

$$V(\mathbf{r}) = -V_1 \frac{y}{d} + V_a(\mathbf{r}) + V_b(\mathbf{r})$$

   where $V_a(\mathbf{r})$ is a solution of part (a) and $-V_1(y/d)$ is a potential of a plane capacitor.

4. Determine the steady-state distribution of the temperature

$$\nabla^2 T(\mathbf{r}) = 0$$

   inside a cylindrical solid, if the constant heat flow q is supplied at the lower end $z = 0$

$$-k \frac{\partial T(\mathbf{r})}{\partial z} = q$$

   while its upper end $z = l$ and the surface $\rho = a$ are maintained at zero temperature.

5. A sphere of radius R, initially at temperature T_0, is cooling off in the environment at absolute zero. Find the temperature at the center of the sphere as a function of time in the limit of long times.

6. For zero initial conditions, find the vibrations of a circular membrane $0 \le r < a$ in a non-resistant medium, produced by the motion of its edge according to

$$u(a,t) = A \sin \omega t, \; 0 < t < \infty$$

   assuming that there are no resonances.

   Hint:Find forced oscillations first, using the following form:

$$U(r,t) = R(r) \sin \omega t$$