Math Physics Test - 11/15/2005

Fourier and Laplace Transforms

1. 20points A circuit, subject to a constant applied voltage V_0,

$$L\ \ddot{Q}\ +R\ \dot{Q}\ +C^{-1}Q = V_0$$

is closed by a switch at $t = 0$. Here Q is the charge on the capacitor (which was initially uncharged). Find the current $I(t) = \dot{Q}$ using two different techniques: 1) by directly solving the differential equation and 2) using the Laplace transform. You may assume that the circuit is under-damped,

$$2\sqrt{L/C} > R$$

Solution

At $t < T$ look for a solution of the homogeneous eq.

$$Q = A \exp st$$

$$Ls^2 + Rs + C^{-1} = 0$$

$$s_{1,2} = -\frac{R}{2L} \pm i\sqrt{\frac{1}{CL} - \left(\frac{R}{2L}\right)^2}$$

so that

$$Q = \exp\left(-\frac{R}{2L}t\right)(A\sin\omega t + B\cos\omega t) + V_0 C$$

where

$$\omega = \sqrt{\frac{1}{CL} - \left(\frac{R}{2L}\right)^2}$$

From $Q(0) = 0$, $B = -V_0 C$ and from $\dot{Q}(0) = 0$

$$-\frac{R}{2L}B + A\omega = 0,\ A = \frac{R}{2L\omega}B$$

and

$$\begin{aligned}
I(t) &= \dot{Q} = \left(-\frac{R}{2L}\right)\exp\left(-\frac{R}{2L}t\right)(A\sin\omega t + B\cos\omega t) + \exp\left(-\frac{R}{2L}t\right)(A\omega\cos\omega t - B\omega\sin\omega t) \\
&= \exp\left(-\frac{R}{2L}t\right)\left[\left(-\frac{R}{2L}A - B\omega\right)\sin\omega t + \left(-\frac{R}{2L}B + A\omega\right)\cos\omega t\right] \\
&= -\exp\left(-\frac{R}{2L}t\right)B\omega\left[\left(\frac{R}{2L\omega}\right)^2 + 1\right]\sin\omega t = -\exp\left(-\frac{R}{2L}t\right)(-V_0 C\omega)\left(CL\omega^2\right)^{-1}\sin\omega t \\
&= \frac{V_0}{L\omega}\exp\left(-\frac{R}{2L}t\right)\sin\omega t
\end{aligned}$$

Using now Laplace transform, $q(s) = \mathcal{L}[Q(t)]$,

$$q(s) = \frac{V_0}{L}\frac{1}{s\left[s^2 + (R/L)s + (1/CL)\right]}$$

and

$$\begin{aligned}
i(s) &= \mathcal{L}\left[\dot{Q}(t)\right] = sq(s) \\
&= \frac{V_0}{L}\frac{1}{s^2 + (R/L)s + (1/CL)}
\end{aligned}$$

whereof

$$\begin{aligned}
I(t) &= \frac{V_0}{L}\frac{1}{2\pi i}\int_C \frac{\exp st}{s^2 + (R/L)s + (1/CL)} = \frac{V_0}{L}\frac{\exp s_1 t - \exp s_2 t}{s_1 - s_2} \\
&= \frac{V_0}{L}\exp\left(-\frac{R}{2L}t\right)\frac{\exp(i\omega) - \exp(-i\omega)}{2i\omega} = \frac{V_0}{L\omega}\exp\left(-\frac{R}{2L}t\right)\sin\omega t
\end{aligned}$$

2. (20points) Evaluate the 3D Fourier transform

$$\mathcal{F}\left[f\left(\mathbf{r}\right)\right]=\int d^3\mathbf{r}f\left(\mathbf{r}\right)\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)$$

of the function

$$f\left(\mathbf{r}\right)=Ar^2\exp\left(-r^2/a^2\right)$$

where

$$A=\frac{4}{a^2\sqrt{15}}\left(\frac{2}{\pi a^2}\right)^{3/4}$$

is the normalization constant, $\int d^3\mathbf{r}f^2\left(\mathbf{r}\right)=1$.

Solution

Denoting $a^{-2}=\alpha$

$$\mathcal{F}\left[f\left(\mathbf{r}\right)\right]=A\int d^3\mathbf{r}r^2\exp\left(-\alpha r^2\right)\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)\equiv-\frac{\partial}{\partial\alpha}I$$

where

$$\begin{aligned}I&=A\int d^3\mathbf{r}\exp\left(-\alpha r^2\right)\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)=2\pi A\int_0^\infty dr\left[r^2\exp\left(-\alpha r^2\right)\int_{-1}^1dz\exp\left(-ikrz\right)\right]\\&=\frac{4\pi A}{k}\int_0^\infty drr\exp\left(-\alpha r^2\right)\sin\left(kr\right)=\frac{2\pi A}{k}\int_{-\infty}^\infty drr\exp\left(-\alpha r^2\right)\sin\left(kr\right)=\\&=\frac{2\pi A}{k}\int_{-\infty}^\infty drr\exp\left(-\alpha r^2\right)\left[\sin\left(kr\right)+i\cos\left(kr\right)\right]=\frac{2\pi Ai}{k}\int_{-\infty}^\infty drr\exp\left(-\alpha r^2-ikr\right)\\&=-\frac{2\pi A}{k}\frac{\partial}{\partial k}\int_{-\infty}^\infty dr\exp\left(-\alpha r^2-ikr\right)=-\frac{2\pi A}{k}\sqrt{\frac{\pi}{\alpha}}\frac{\partial}{\partial k}\exp\left(-k^2/4\alpha\right)\\&=\frac{2\pi A}{k}\sqrt{\frac{\pi}{\alpha}}\frac{2k}{4\alpha^2}\exp\left(-k^2/4\alpha\right)=A\left(\frac{\pi}{\alpha}\right)^{3/2}\exp\left(-k^2/4\alpha\right)\end{aligned}$$

so thatw

$$\begin{aligned}\mathcal{F}\left[f\left(\mathbf{r}\right)\right]&=-A\exp\left(-k^2/4\alpha^2\right)\left(\frac{\partial}{\partial\alpha}\left(\frac{\pi}{\alpha}\right)^{3/2}+\left(\frac{\pi}{\alpha}\right)^{3/2}\frac{k^2}{4\alpha^2}\right)=A\exp\left(-k^2/4\alpha^2\right)\frac{\pi^{3/2}}{\alpha^{5/2}}\left(\frac{3}{2}-\frac{k^2}{4\alpha}\right)\\&=\frac{4\left(2\pi a^2\right)^{3/4}}{\sqrt{15}}\left(\frac{3}{2}-\frac{k^2a^2}{4}\right)\exp\left(-k^2/4\alpha^2\right)=\frac{\left(2\pi a^2\right)^{3/4}}{\sqrt{15}}\left(6-k^2a^2\right)\exp\left(-k^2/4\alpha^2\right)\end{aligned}$$