Math Physics Test - 11/01/2005

Integration

1. Evaluate

$$I = \int_0^\infty \frac{x^{1/4}}{x^2+1} dz$$

Solution

Consider the integral

$$\widetilde{I} = \oint_C \frac{z^{1/4}}{z^2+1} dz$$

where C is the closed contour which consists of a straight line along the upper bank of the real axis and an infinite semicircle in the upper plane.

$$\widetilde{I} = I + \int_\infty^0 \frac{x^{1/4}\exp(i\pi/4)}{x^2+1}\exp(i\pi)\,dx = I\left(1+\exp\left(\frac{i\pi}{4}\right)\right)$$
$$= 2\pi i \frac{z^{1/4}}{z+i}\mid_{z=i} = \pi\exp\left(\frac{i\pi}{8}\right)$$

$$I = \frac{\pi}{2}\sec\left(\frac{\pi}{8}\right) = \frac{\pi}{\sqrt{\sqrt{2}+2}}$$

2. Find the asymptotic behavior of the function

$$f(x) = \int_0^x \frac{\sin t^2 dt}{t^2}$$

in the limits $x \to \infty$ and $x \to 0$. (For both limits, keep the first two terms in the expansion.)

Hint: You may need to evaluate

$$I = \int_0^\infty \frac{\sin t^2 dt}{t^2}$$

Solution

For $x \to \infty$, integrating by parts,

$$\begin{aligned} f(x) &= \int_0^\infty \frac{\sin t^2 dt}{t^2} - \int_x^\infty \frac{\sin t^2 dt}{t^2} = \sqrt{\frac{\pi}{2}} + \int_x^\infty \frac{d(\cos t^2)}{2t^3} = \sqrt{\frac{\pi}{2}} + \frac{\cos t^2}{2t^3} |_x^\infty + \int_x^\infty \frac{3\cos t^2 dt}{2t^4} \\ &= \sqrt{\frac{\pi}{2}} - \frac{\cos x^2}{2x^3} + \dots \end{aligned}$$

For $x \to 0$, expanding,

$$f(x) = \int_0^x \frac{(t^2 - t^6/6 + \dots)\, dt}{t^2} = x - x^5/30 + O\left(x^9\right)$$

To evaluate the integral,

$$I(\alpha) = \int_0^\infty \frac{\exp(-\alpha t^2) \sin t^2 dt}{t^2};\ I(0) = I,\ I(\infty) = 0$$

$$\begin{aligned} \frac{dI(\alpha)}{d\alpha} &= -\int_0^\infty \exp\left(-\alpha t^2\right) \sin t^2 dt = -\frac{1}{2i} \int_0^\infty \exp\left(-\alpha t^2\right) \left[\exp\left(it^2\right) - c.c.\right] dt \\ &= -\frac{1}{2i} \frac{\sqrt{\pi}}{2} \left(\frac{1}{\sqrt{\alpha - i}} - \frac{1}{\sqrt{\alpha + i}}\right) = -\frac{1}{2i} \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\alpha^2 + 1}} \left(\exp\left(\frac{1}{2} i \tan^{-1} \alpha\right) - c.c.\right) \\ &= -\frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\alpha^2 + 1}} \sin\left(\frac{1}{2} \tan^{-1} \alpha\right) \end{aligned}$$

$$I(\alpha) = -\frac{\sqrt{\pi}}{2} \int_\infty^\alpha \frac{1}{\sqrt{\alpha^2 + 1}} \sin\left(\frac{1}{2} \tan^{-1} \alpha\right)$$

$$\begin{aligned} I &= -\frac{\sqrt{\pi}}{2} \int_\infty^0 \frac{1}{\sqrt{\alpha^2 + 1}} \sin\left(\frac{1}{2} \tan^{-1} \alpha\right) \overset{x = \left(\tan^{-1} \alpha\right)/2}{=} \sqrt{\pi} \int_0^{\pi/4} \frac{dx \sin x \sqrt{\sin 2x}}{\sin^2 2x} \\ &= \sqrt{\pi} \int_0^{\pi/4} \frac{dx}{\sin 2x} \sqrt{\frac{\sin^2 x}{\sin 2x}} = \sqrt{\frac{\pi}{2}} \int_0^{\pi/4} \frac{dx}{\sin 2x} \sqrt{\frac{\sin x}{\cos x}} \overset{y^2 = \tan x}{=} \sqrt{\frac{\pi}{2}} \int_0^1 dy = \sqrt{\frac{\pi}{2}} \end{aligned}$$

3. Evaluate

$$I = \int \frac{1}{\left(1 + \mathbf{a}\cdot\widehat{\mathbf{r}} + \mathbf{b}\cdot\widehat{\mathbf{r}}\right)^2} d\Omega$$

Solution

Introducing $\mathbf{c} = \mathbf{a} + \mathbf{b}$,

$$\begin{aligned} I &= \int \frac{1}{\left(1 + \mathbf{c} \cdot \widehat{\mathbf{r}}\right)^2} d\Omega = 2\pi \int_{-1}^{1} \frac{1}{\left(1 + cz\right)^2} dz \\ &= \frac{2\pi}{c} \left(\frac{1}{1-c} - \frac{1}{1+c} \right) = \frac{4\pi}{1-c^2} = \frac{4\pi}{1 - a^2 - b^2 - 2\mathbf{a} \cdot \mathbf{b}} \end{aligned}$$