# Math Physics Test - 10/13/2005

Ordinary Differential Equations

1. Solve the differential equation

$$y'\left(\frac{x}{y} - y\sin y\right) - 1 = 0$$

*Solution*

Treat $x$ as a function of $y$, $x = x\,(y)$

$$\frac{x}{y} - \frac{dx}{dy} - y\sin y = 0$$

$$-y\left(\frac{x}{y}\right)' - y\sin y = 0$$

$$x = y\,(C + \cos y)$$

2. A harmonic oscillator, initially at rest, is driven by a constant force for one half period of free oscillations:

$$\ddot{x} + \omega^2 x = \frac{F}{m} w\,(t)$$

where

$$w\,(t) = \begin{matrix} 1 & 0 < t < \pi/\omega \\ 0 & t > \pi/\omega \end{matrix}$$

(a) Find the motion of the oscillator for $0 < t < \pi/\omega$

(b) (**bonus**) Find the motion for $t > \pi/\omega$

*Solution*

Using variational technique (don't need to here - just for practice):

$$x = u_1 \sin \omega t + u_2 \cos \omega t$$

$$\begin{aligned} \dot{u}_1 \sin \omega t + \dot{u}_2 \cos \omega t &= 0 \\ \dot{u}_1 \cos \omega t - \dot{u}_2 \sin \omega t &= \frac{F}{m\omega} \end{aligned}$$

For $t < \pi/\omega$,

$$\begin{aligned} \dot{u}_1 &= \frac{F}{\omega m} w(t) \cos \omega t \Rightarrow u_1 = \frac{F}{\omega m} \int_0^t \cos \omega t dt = \frac{F}{\omega^2 m} \sin \omega t \\ \dot{u}_2 &= -\frac{F}{\omega m} w(t) \sin \omega t \Rightarrow u_2 = -\frac{F}{\omega m} \int_0^\infty \sin \omega t dt = \frac{F}{\omega^2 m} (\cos \omega t - 1) \end{aligned}$$

The full motion

$$x = A \sin \omega t + B \cos \omega t + \frac{F}{\omega^2 m}$$

From $x(0) = 0$, $B = -2F/\omega^2 m$, from $\dot{x}(0) = 0$, $A = 0$

$$x = \frac{F}{\omega^2 m} (1 - \cos \omega t)$$

Notice that

$$x(\pi/\omega) = 2F/\omega^2 m, \dot{x}(\pi/\omega) = 0$$

For $t > \pi/\omega$,

$$\begin{aligned} \dot{u}_1 &= \frac{F}{\omega m} w(t) \cos \omega t \Rightarrow u_1 = \frac{F}{\omega m} \int_0^{\pi/\omega} \cos \omega t dt = 0 \\ \dot{u}_2 &= -\frac{F}{\omega m} w(t) \sin \omega t \Rightarrow u_2 = -\frac{F}{\omega m} \int_0^{\pi/\omega} \sin \omega t dt = -\frac{2F}{\omega^2 m} \end{aligned}$$

The full motion

$$x = -\frac{2F}{\omega^2 m} \cos \omega t$$

3. The Chebyshev polynomials $T_n(x)$ are solutions of the equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0$$

Show that $n$ must be integer for convergency at $\infty$ and find $T_2(x)$ and $T_3(x)$ (up to a constant multiplier).

*Solution*

Look for a solution as

$$y=\sum_{n=0}^{\infty}c_m x^m$$

The recursion relationship

$$(m+1)(m+2)\,c_{m+2}=\left(m^2-n^2\right)c_m$$

so that

$$\frac{c_{m+2}}{c_m}=\frac{m^2-n^2}{(m+1)(m+2)}\overset{m\to\infty}{\longrightarrow}1$$

and the series diverges unless $n=m$ - integer. There are even and odd series. Setting $c_0=1$ when $c_1=0$ and vice versa. Using recursion relationship

$$T_2=1-2x^2$$
$$T_3=x-\frac{4}{3}x^3$$