Mathematical Physics Final Exam
Boundary Value Problems, Green's Functions
12/06/2005

1. Find the two dimensional potential $V(r,\theta)$ inside a circle of radius a with no charges,

$$\bigtriangledown^2 V = 0$$

for the boundary potential given by

$$V(a,\theta) = V_0 \sin(\theta)$$

Hint:

$$\bigtriangledown^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Solution

Look for the solution in the form

$$V = R(r)\,\Theta(\theta)$$

then

$$\frac{1}{R}r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = 0$$

so that

$$\frac{1}{R}r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = n^2$$

whereof

$$\Theta = \begin{matrix}\sin n\theta \\ \cos n\theta\end{matrix}$$

Because of the boundary condition $n = 1$ and $\Theta = \sin\theta$ and

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - R = 0$$

Look for a solution $R \propto r^\alpha$ and find

$$\alpha^2 - 1 = 0,\ \alpha = 1 \text{ (convergence at } r = 0\text{)}$$

so that

$$V = Ar\sin\theta = V_0\frac{r}{a}\sin\theta$$

2. The two surfaces of an infinite heat conducting slab of thickness D are in contact with the thermal baths at temperatures T_0 ($x = 0$) and zero ($x = D$) respectively. Find the temperature inside the slab for $t \gg D^2/\kappa$, if initially the slab's temperature is T_0.

Hint:

$$\left(\frac{\partial}{\partial t} - \kappa\frac{\partial^2}{\partial x^2}\right)T = 0$$

and look for a solution in the form $T = T_1(x) + \triangle T(x,t)$, where $T_1(x)$ satisfies the boundary conditions and $\triangle T(x,t)$ has zero b.c.

Solution

Look for a solution $T = T_1(x) + \triangle T(x,t)$ where $T_1(x)$ satisfies boundary condition so that

$$T_1(x) = T_0\left(1 - \frac{x}{D}\right)$$

and $\triangle T \propto e^{-\alpha t} X(x)$,

$$X_n = A_n \sin\left(\sqrt{\frac{\alpha_n}{\kappa}} x\right) = A_n \sin\left(\frac{n\pi x}{D}\right)$$

where

$$\sqrt{\frac{\alpha_n}{\kappa}} D = n\pi$$

Consequently

$$T = T_0\left(1 - \frac{x}{D}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{D}\right) \exp\left(-\alpha_n t\right)$$

From the initial condition

$$T_0 = T_0\left(1 - \frac{x}{D}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{D}\right)$$

and

$$A_n = \frac{2T_0}{D^2} \int_0^D x \sin\left(\frac{n\pi x}{D}\right) dx = -2T_0 \frac{\cos n\pi}{n\pi} = 2T_0 \frac{(-1)^{n+1}}{n\pi}$$

Finally, for $t \gg D^2/\kappa$,

$$T = T_0\left[1 - \frac{x}{D} + \frac{2}{\pi} \sin\left(\frac{\pi x}{D}\right) \exp\left(-\frac{\pi^2 \kappa t}{D^2}\right)\right]$$

3. The Poisson's equation in one dimension has the following form:

$$\frac{d^2 V}{dx^2} = 2\varrho(x)$$

where $\varrho(x)$ is a 1D charge density. Write the solution of the Poisson's equation in terms of the Green's function $G(x - x')$ of the Laplace operator and, in particular, find the potential at origin given

$$\varrho(x) = \begin{array}{ll} q/\ell, & -\ell \leq x \leq \ell \\ 0, & |x| > \ell \end{array}$$

Hint: $G(x - x') = G(|x - x'|)$ can be found by taking Fourier transform of

$$\frac{d^2}{dx^2} G(x) = \delta(x)$$

and treating the pole on the axis of integration as a principal value.

Solution

Taking FT, find

$$-k^2 G(k) = \frac{1}{2\pi}$$

and

$$G(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{\exp(ikx)}{k^2}$$

so that

$$G(x) = \begin{array}{l} -i\left[\operatorname{res}\left(\exp(ikx)/k^2,\ k=0\right)/2\right] = -(i/2)\, d\exp(ikx)/dk|_{k=0} = x/2,\ x > 0 \\ i\left[\operatorname{res}\left(\exp(ikx)/k^2,\ k=0\right)/2\right] = (i/2)\, d\exp(ikx)/dk|_{k=0} = -x/2,\ x < 0 \end{array} = \frac{|x|}{2}$$

Consequently,

$$V(x) = \int_{-\infty}^{\infty} dx' \left|x - x'\right| \varrho(x')$$

For the given $\varrho(x)$, the potential at the origin is

$$V(0) = 2\frac{q}{\ell} \int_0^{\ell} dx' x' = q\ell$$