Math Physics Test - 11/16/2004

Fourier and Laplace Transforms

1. A harmonic oscillator, initially at rest, is driven by the force F_0 which acts for a finite time T. Using the Laplace transform method, find the amplitude of the resulting oscillations. *Bonus*: Interpret your result in the limit $\omega T \ll 1$.

   Solution

$$\ddot{x} + \omega^2 x = \frac{F}{m}$$

$$\left(s^2 + \omega^2\right) X = \frac{F_0}{m} \int_0^T \exp(-st)\, dt = \frac{F_0}{ms} \left[1 - \exp(-sT)\right]$$

$$X = \frac{F_0}{m} \frac{\left[1 - \exp(-sT)\right]}{s\left(s^2 + \omega^2\right)}$$

$$\begin{aligned} x &= \frac{F_0}{m} \oint_C \frac{\left[1 - \exp(-sT)\right]}{s\left(s^2 + \omega^2\right)} \exp(st)\, ds \\ &= \frac{F_0}{m} \left\{ \frac{\left[1 - \exp(-i\omega T)\right] \exp(i\omega t)}{(2i\omega)\, i\omega} + \frac{\left[1 - \exp(i\omega T)\right] \exp(-i\omega t)}{(-2i\omega)(-i\omega)} \right\} \\ &= \frac{F_0}{m\omega^2} \left[-\cos(\omega t) + \cos(\omega(t - T))\right] = 2\frac{F_0}{m\omega^2} \sin\left(\frac{\omega T}{2}\right) \sin\left(\omega t + \frac{\omega T}{2}\right) \end{aligned}$$

   Therefore the amplitude is

$$a = 2\frac{F_0}{m\omega^2} \sin\left(\frac{\omega T}{2}\right) \overset{\omega T \ll 1}{\rightarrow} \frac{F_0 T}{m\omega} = \frac{P}{m\omega}$$

   where P is the impulse given to the oscillator.

2. Evaluate the Fourier transform $g(\omega)$ of the force from the previous problem. *Bonus*: Confirm that $\int_{-\infty}^{\infty} |F(t)|^2 \, dt = (2\pi)^{-1} \int_{-\infty}^{\infty} |g(\omega)|^2 \, d\omega$ by direct evaluation of the integrals.

Solution

$$\begin{aligned} g(\omega) &= \int_{-\infty}^{\infty} F(t) \exp(-i\omega t) \, dt = \int_0^T F_0 \exp(-i\omega t) \, dt \\ &= \frac{F_0}{i\omega} [1 - \exp(-i\omega T)] = F_0 \exp\left(-i\frac{\omega T}{2}\right) \frac{2 \sin(\omega T/2)}{\omega} \end{aligned}$$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(\omega)|^2 \, d\omega &= \frac{1}{2\pi} (2F_0)^2 \int_{-\infty}^{\infty} \frac{\sin^2(\omega T/2)}{\omega^2} d\omega \\ &= \frac{2F_0^2}{\pi} \frac{T}{2} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = F_0^2 T = \int_0^T F_0^2 dt = \int_{-\infty}^{\infty} |F(t)|^2 \, dt \end{aligned}$$