Math Physics Test - 11/02/2004

Integration

1. Evaluate integral (all calculations must be done from first principles):

$$I = -\frac{1}{(2\pi)^3} \int \frac{\exp\left(i\overrightarrow{k} \cdot \overrightarrow{r}\right)}{k^2} d^3\overrightarrow{k}$$

Solution

$$\begin{aligned}
I &= -\frac{1}{(2\pi)^2} \int_0^\infty \int_{-1}^1 \exp(ikrz)\, dz dk \\
&= -\frac{1}{(2\pi)^2} \int_0^\infty \frac{\exp(ikr) - \exp(-ikr)}{ikr} dk \\
&= -\frac{1}{2(2\pi)^2 ri} \int_{-\infty}^\infty \frac{\exp(ix) - \exp(-ix)}{x} dx \\
&= -\frac{2\pi i}{2(2\pi)^2 ri} = -\frac{1}{4\pi r}
\end{aligned}$$

2. Give the first two terms of expansion of

$$\operatorname{Si} x = \int_0^x \frac{\sin t}{t} dt$$

for $x \ll 1$ and $x \gg 1$.

Solution

$$\operatorname{Si} x \approx \int_0^x \frac{t - t^3/6}{t} dt = x - \frac{x^3}{18} + O\left(x^3\right)$$

$$\operatorname{Si} x \approx \frac{\pi}{2} - \int_x^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} - \frac{\cos x}{x} + O\left(x^{-2}\right)$$

3. Evaluate integral,$-1 < \alpha < 3$,

$$I = \int \frac{x^\alpha}{x^4+1} dx$$

Solution

Consider the integral

$$J = \oint_C \frac{z^\alpha}{z^4+1} dz$$

where the closed contour C consists of a real axis from 0 to $+\infty$, one quarter of a large circle at $|z| = \infty$, and a return to the origin along the imaginary axis, $\arg z = \pi/2$. This contour encloses a simple pole at $\exp(i\pi/4)$ and so

$$J = I\left\{1 - \exp\left[\frac{\hat{\imath}\pi}{2}(\alpha+1)\right]\right\} = 2\pi i\left[\frac{z^\alpha}{4z^3}\right]_{z=\exp(i\pi/4)} = \frac{\pi i}{2}\exp\left(\frac{i\pi}{4}(\alpha-3)\right)$$

$$I\exp\left[\frac{\hat{\imath}\pi}{4}(\alpha+1)\right]\left\{\exp\left[-\frac{\hat{\imath}\pi}{4}(\alpha+1)\right] - \exp\left[\frac{\hat{\imath}\pi}{4}(\alpha+1)\right]\right\} = \frac{\pi i}{2}\exp\left(\frac{i\pi}{4}(\alpha-3)\right)$$

$$-2iI\sin\left[\frac{\pi}{4}(\alpha+1)\right] = \frac{\pi i}{2}\exp(-i\pi) = -\frac{\pi i}{2}$$

$$I = \frac{\pi}{4\sin\left[\frac{\pi}{4}(\alpha+1)\right]}$$