# Math Physics Test - 10/12/2004

Ordinary Differential Equations

1. Find the general solution of the differential equation

$$xy'' + 2y' - xy = x^2$$

*Solution*

$$xy'' + 2y' - xy = x^2$$

$$(xy)'' - (xy) = x^2$$

Complimentary solution ($z = xy$)

$$z'' - z = 0$$

$$z = c_1 e^x + c_2 e^{-x}$$

Particular solution

$$z = -x^2 - 2$$

Combining the two,

$$y = \frac{1}{x}\left(c_1 e^x + c_2 e^{-x} - 2\right) - x$$

2. A harmonic oscillator, initially at rest, is driven by the force $F_0 \exp(-\lambda t)$. Find the amplitude of the resulting oscillations. *Bonus*: Interpret your result in the limit $\lambda \gg \omega$.

*Solution*

Complimentary solution

$$x = A \sin \omega t + B \cos \omega t$$

Particular solution

$$x = \frac{F_0}{m(\lambda^2 + \omega^2)} \exp(-\lambda t)$$

General solution

$$x = A \sin \omega t + B \cos \omega t + \frac{F_0}{m(\lambda^2 + \omega^2)} \exp(-\lambda t)$$

From $x(0) = 0$, $\dot{x}(0) = 0$

$$x = \frac{F_0}{m(\lambda^2 + \omega^2)} \left( \frac{\lambda}{\omega} \sin \omega t - \cos \omega t + \exp(-\lambda t) \right)$$

The amplitude of oscillations

$$x_m = \frac{F_0}{m\sqrt{\lambda^2 + \omega^2}\omega}$$

For $\lambda \gg \omega$, the force acts during a time interval much shorter than the period of oscillations and its impulse

$$P = \int_0^\infty dt F_0 \exp(-\lambda t) = F_0/\lambda$$

determines the amplitude of the oscillations

$$x_m = \frac{v_m}{\omega} \approx \frac{(P/m)}{\omega} = \frac{F_0}{m\lambda\omega}$$