Math Physics Final - 12/09/2004

Boundary Value Problems

1. Consider a string of length ℓ stretched along the x-axis and subject to a transverse displacement $u(x,t)$,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

The string is clamped at one end and is attached at the other end to a massless ring which slides freely on a fixed rod

$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(\ell,t) = 0$$

Assuming that

$$u(x,0) = \sin\frac{\pi x}{2\ell},\ \frac{\partial u}{\partial t}(x,0) = 0$$

Find solution for the displacement $u(x,t)$.

Solution

Look for a solution $u(x,t) = X(x)\{\sin\omega t, \cos\omega t\}$,

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) X = 0,\ k = \frac{\omega}{v}$$

Look for a solution $X = A\sin kx + B\cos kx$, subject to b.c.

$$X(0) = 0,\ \frac{\partial X}{\partial x}(\ell) = 0$$

Therefore,

$$X_n \propto \sin k_n x,\ k_n = \frac{(2n+1)\pi}{2\ell}$$

and

$$u(x,t) = \sum_{n=0}^{\infty} (C_n \sin \omega_n t + D_n \cos \omega_n t) \sin k_n x$$

From i.c. $\partial u / \partial t\,(x,0) = 0$, $C_n = 0$ and

$$u(x,t) = \sum_{n=0}^{\infty} D_n \cos \omega_n t \sin k_n x$$

where, from the other initial condition,

$$\sum_{n=0}^{\infty} D_n \sin k_n x = \sin \frac{\pi x}{2\ell}$$

Therefore $D_0 = 1$, $D_{n \neq 0} = 0$ and

$$u(x,t) = \cos \frac{\pi v t}{2\ell} \sin \frac{\pi x}{2\ell}$$

2. The boundary of a thin circular disk of radius a is maintained at temperature

$$T_0 (1 + \alpha \sin \theta), \; 0 \leq \theta \leq 2\pi, \; 0 \leq \alpha \leq 1$$

Find the steady state temperature field

$$\nabla^2 T = \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} \right)^2 \right] T = 0$$

Solution

Look for solution as $T = T_0 + \triangle T$, where $\triangle T = R(r) \{\sin(m\theta), \cos(m\theta)\}$

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) - \frac{m^2}{r^2} \right] R = 0$$

Look for a solution $R \propto r^b$ and find $b = m > 0$ so that

$$\triangle T = \sum_{m=0}^{\infty} r^m (A_m \sin m\theta + B_m \cos m\theta)$$

Using b.c.

$$T_0\alpha\sin\theta = \sum_{m=0}^{\infty} a^m \left(A_m \sin m\theta + B_m \cos m\theta\right)$$

whereof $A_{m\neq 1}, B_m = 0,\ A_1 = T_0\alpha/a$ and

$$T = T_0\left(1 + \alpha\frac{r}{a}\sin\theta\right)$$

3. **Bonus problem**. The wave function of a three-dimensional rotator is given by

$$\psi = A\cos^2\theta$$

What are the possible values of l? What are their probabilities?

Hint: expand ψ in terms of the spherical functions $Y_{l0}(\cos\theta) \propto P_l(\cos\theta)$.

Some useful formulae: $P_0(\cos\theta) = 1,\ P_1(\cos\theta) = \cos\theta,\ P_2(\cos\theta) = (3\cos^2\theta - 1)/2,$ $P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2,\ \ldots$

Solution

Only $l = 0, 2$ are possible . From normalization

$$2\pi\int_0^{\pi} Y_{l0}^2(\cos\theta)\sin\theta d\theta = 1$$

find

$$Y_{00} = \frac{1}{\sqrt{4\pi}}P_0(\cos\theta)\,,\ Y_{20} = \sqrt{\frac{5}{4\pi}}P_2(\cos\theta)$$

and

$$\psi = A\frac{\sqrt{4\pi}}{3}\left[Y_{00} + \frac{2}{\sqrt{5}}Y_{20}\right]$$

so that the probability to have $l = 0$ is 5/9 and $l = 2$ is 4/9.