MathPhys - Winter 2003
Quiz 5

1. (20 points) Evaluate the integral

$$\int_{-1}^{1} P_l\,(x)\,dx$$

using two different methods
(a) Use

$$\frac{1}{\sqrt{1-2hx+h^2}} = \sum_{n=0}^{\infty} h^n P_n\,(x)$$

to evaluate $P_n\,(1)$ and $P_n\,(-1)$ and to prove that

$$P_n = \frac{P'_{n+1} - P'_{n-1}}{(2n+1)}$$

and
(b) Use the orthonormality

$$\int_{-1}^{1} P_n\,(x)\,P_m\,(x)\,dx = \frac{2}{2n+1}\delta_{nm}$$

Solution

$$\begin{aligned}\frac{1}{\sqrt{1-2h+h^2}} &= \frac{1}{1-h} = \sum_{n=0}^{\infty} h^n P_n\,(1)\\ P_n\,(1) &= 1\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{1+2h+h^2}} &= \frac{1}{1+h} = \sum_{n=0}^{\infty} h^n P_n\,(-1)\\ P_n\,(-1) &= (-1)^n\end{aligned}$$

Differentiating on h,

$$nP_n - (2n-1)\,xP_{n-1} + (n-1)\,P_{n-2} = 0$$

Further differentiation gives

$$nP'_n - (2n-1)\,P_{n-1} - (2n-1)\,xP'_{n-1} + (n-1)\,P'_{n-2} = 0$$

Differentiating on x, and multiplying by n

$$nP'_n - 2xnP'_{n-1} + nP'_{n-2} - nP_{n-1} = 0$$

Subtracting the last two equations

$$-2xnP'_{n-1} - nP_{n-1} + (2n-1)\,P_{n-1} + (2n-1)\,xP'_{n-1} + P'_{n-2} = 0$$

or

$$-xP'_{n-1} + (n-1) P_{n-1} + P'_{n-2} = 0$$

and, finally,

$$xP'_n - P'_{n-1} = nP_n$$

Similarly,

$$-xP'_n + P'_{n+1} = (n+1) P_n$$

So that

$$P_n = \frac{P'_{n+1} - P'_{n-1}}{(2n+1)}$$

and

$$\int_{-1}^{1} P_l(x)\, dx = \frac{P_{l+1}(1) - P_{l-1}(1)}{(2l+1)} - \frac{P_{l+1}(-1) - P_{l-1}(-1)}{(2l+1)} = \frac{2}{2l+1}\delta_{l0}$$

Alternatively,

$$\int_{-1}^{1} P_l(x) P_0(x)\, dx = \frac{2}{2l+1}\delta_{l0}$$

2. (15 points) Using the generating function for Hermit polynomials,

$$\exp\left(2hx - h^2\right) = \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!}$$

evaluate

$$\frac{d^n H_n(x)}{dx^n}$$

and

$$H_n(0)$$

Solution

$$\frac{d^n H_n(x)}{dx^n} = \frac{d^n}{dx^n}\left[\frac{d^n}{dh^n}\exp\left(2hx - h^2\right)_{h=0}\right] = \frac{d^n}{dh^n}\left[\frac{d^n \exp\left(2hx - h^2\right)}{dx^n}\right]_{h=0}$$
$$= 2^n \frac{d^n}{dh^n}\left[h^n \exp\left(2hx - h^2\right)\right]_{h=0} = 2^n n!$$

and

$$\exp\left(-h^2\right) = \sum_{n=0}^{\infty} H_n(0) \frac{h^n}{n!}$$

whereof

$$H_{2m+1}(0) = 0,\ H_{2m}(0) = (-1)^m \frac{(2m)!}{m!}$$

3. (15 points) The integral representation for the Bessel's function is

$$J_n\left(z\right) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp\left[i\left(z\sin\theta - n\theta\right)\right]$$

Recall that Fourier series of a 2π-periodic can be written as

$$f\left(\theta\right) = \sum_{n=-\infty}^{\infty} a_n \exp\left(in\theta\right)$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f\left(\theta\right)\exp\left(-in\theta\right)$$

Use the latter to derive the generating function for Bessel's functions.

Solution

$$\exp\left(iz\sin\theta\right) = \sum_{n=-\infty}^{\infty} J_n\left(z\right)\exp\left(in\theta\right) = \sum_{n=-\infty}^{\infty} J_n\left(z\right)\left[\exp\left(i\theta\right)\right]^n$$

Introducing $h = \exp\left(i\theta\right)$ and

$$i\sin\theta = \frac{1}{2}\left(h - \frac{1}{h}\right)$$

we find

$$\exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n\left(z\right)h^n$$