MathPhys - Fall 2002
Quiz 3

1. (6 points) Evaluate the integrals

$$I(\mathbf{k}) = \int d\Omega\,(\mathbf{k}\cdot\mathbf{r})$$

   (a)

$$\Phi(\mathbf{a},\mathbf{b},\mathbf{c}) = \int d\Omega\,(\mathbf{a}\cdot\mathbf{r})\,(\mathbf{b}\cdot\mathbf{r})\,(\mathbf{c}\cdot\mathbf{r})$$

   (b)

   Solution

   Both are zero for symmetry reasons

2. (8 points) Evaluate the integral

$$I = \int_0^\infty \frac{\sqrt{x}}{(x+1)^2}dx$$

   Solution

   Using the usual contour

$$\widetilde{I} = 2I = 2\pi i \frac{d}{dz}\sqrt{z}\,|_{z=e^{i\pi}} = \frac{\pi i}{\sqrt{e^{i\pi}}} = \pi$$

$$I = \frac{\pi}{2}$$

   Alternatively

$$I = \int_0^\infty \frac{\sqrt{x}}{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2}\frac{dx}{x} = \int_0^\infty \frac{2\sqrt{x}}{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2}\frac{d\sqrt{x}}{\sqrt{x}} = \int_0^\infty \frac{2dz}{(z+z^{-1})^2}$$
$$= 2\left(\int_0^\infty \frac{dz}{(z^2+1)} - \int_0^\infty \frac{dz}{(z^2+1)^2}\right) = 2\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

3. (8 points) Evaluate the integral

$$I = \int_0^\infty \frac{x\sin x}{1+x^2}dx$$

   Solution

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx = \frac{1}{4}\int_{-\infty}^\infty \frac{\exp(iax)+\exp(-iax)}{1+x^2}dx = \frac{1}{4}\left(\oint_{upper}\frac{\exp(iaz)}{1+z^2} + \oint_{lower}\frac{\exp(-iaz)}{1+z^2}\right)$$
$$= \frac{2\pi i}{4}\left(\frac{\exp(-a)}{2i} - \frac{\exp(-a)}{-2i}\right) = \frac{\pi}{2}\exp(-a)$$

$$I = -\frac{\partial}{\partial a}\int_0^\infty \frac{\cos ax}{1+x^2}dx \,|_{a=1} - \frac{\partial}{\partial a}\frac{\pi}{2}\exp(-a)\,|_{a=1} = \frac{\pi}{2e}$$

Alternatively,

$$I = \int_0^\infty \frac{x\sin x}{1+x^2}dx = \frac{1}{4i}\left(\oint_{upper}\frac{z\exp(iz)}{1+z^2} - \oint_{lower}\frac{z\exp(-iz)}{1+z^2}\right)$$
$$= \frac{2\pi i}{4i}\left(\frac{i\exp(-1)}{2i} + \frac{(-i)\exp(-1)}{-2i}\right) = \frac{\pi}{2}\exp(-1) = \frac{\pi}{2e}$$

4. (8 points) The sine integral is defined as

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt$$

Find

(a) The first two terms in the series expansion for $x \ll 1$
Solution

$$\mathrm{Si}(x) \approx \int_0^x \frac{t - t^3/6}{t}dt = x - \frac{x^2}{18}$$

(b) Three terms in the asymptotic expansions for $x \to \infty$
Solution

$$\mathrm{Si}(x) = \int_0^\infty \frac{\sin t}{t}dt - \int_x^\infty \frac{\sin t}{t}dt = \frac{\pi}{2} - \frac{\cos x}{x} + \int_x^\infty \frac{\cos t}{t^2}dt = \frac{\pi}{2} - \frac{\cos x}{x} - \frac{\sin x}{x^2} - ...$$

5. (10 points - bonus) Evaluate the integral

$$I = \int_0^\pi \frac{\cos 2x}{a\cos x + 1}dx,\ a^2 < 1$$

Solution

$$I = \frac{1}{2}\int_0^{2\pi}\frac{2\cos^2 x - 1}{a\cos x + 1}dx = \int_0^{2\pi}\frac{\cos^2 x}{a\cos x + 1}dx - \frac{1}{2}\int_0^{2\pi}\frac{1}{a\cos x+1}dx \equiv I_1 - I_2$$

$$I_2 = \frac{1}{i}\oint \frac{1}{a(z+z^{-1})+2}\frac{dz}{z} = \frac{1}{i}\oint\frac{1}{a(z^2+1)+2z}dz = \frac{1}{ia}\oint\frac{1}{(z^2+1)+2z/a}dz =$$
$$= \frac{2\pi i}{ia}\frac{1}{2\sqrt{a^{-2}-1}} = \frac{\pi}{\sqrt{1-a^2}}$$

$$I_1 = \frac{1}{a^2}\int_0^{2\pi}\frac{a^2\cos^2 x}{a\cos x+1}dx = \frac{1}{a^2}\int_0^{2\pi}\frac{a^2\cos^2 x - 1}{a\cos x + 1}dx + \frac{1}{a^2}\int_0^{2\pi}\frac{1}{a\cos x+1}dx$$
$$= \frac{1}{a^2}\int_0^{2\pi}(a\cos x - 1)\,dx + \frac{2}{a^2}I_2 = \frac{1}{a^2}(2I_2 - 2\pi) = \frac{2\pi}{a^2}\left(\frac{1}{\sqrt{1-a^2}} - 1\right)$$

$$I = \frac{2\pi}{a^2}\left(\frac{1}{\sqrt{1-a^2}} - 1\right) - \frac{\pi}{\sqrt{1-a^2}} = \frac{\pi}{\sqrt{1-a^2}}\left(1 - \sqrt{1-a^2}\right)^2$$