MathPhys - Fall 2002
Quiz 1

1. (15 points - see breakdown below) Consider the dimensionless (in atomic units) Schrödinger equation for the radial part of the bound wave function in the Coulomb field:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)R}{r^2} + 2\left(E + \frac{1}{r}\right)R = 0$$

where $E < 0$ and $l \geq 0$ is the integer quantum number (angular momentum).

   (a) (3 points) Find the asymptotic behavior of $R(r)$ at $r \to 0$.
   Solution

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)R}{r^2} = 0$$

$$R \propto r^\alpha$$

$$\alpha(\alpha - 1) + 2\alpha - l(l+1) = 0$$

$$\alpha = l \text{ (reject divergent solution with } \alpha = -l-1)$$

   (b) (3 points) Find the asymptotic behavior of $R(r)$ at $r \to \infty$.
   Solution

$$\frac{d^2R}{dr^2} + 2ER = 0$$

$$R = \exp\left(-\sqrt{-2E}r\right) \text{ (reject divergent solution } \exp\left(\sqrt{-2E}r\right))$$

   (c) (5 points) Make a change of variable, reflecting the above asymptotic behaviors, and find a power series solution for the new variable.
   Solution
   Introduce $\sqrt{-2E} = 1/n$, $\rho = 2r/n$

$$R = \rho^l \exp(-\rho/2)\, w(\rho)$$

$$\rho w''(\rho) + (2l + 2 - \rho)w' + (n - l - 1)w = 0$$

$$w = \sum_{m=0}^{\infty} c_m \rho^m$$

$$[m(m-1) + 2l + 2]\, c_m + [-m + (n - l - 1)]\, c_{m-1} = 0$$

$$\frac{c_m}{c_{m-1}} = \frac{m + l + 1 - n}{m(m-1) + 2l + 2}$$

(d) (4 points) Derive the condition of the series convergence and the energies of the bound states.

Solution

$$\frac{c_m}{c_{m-1}} = \frac{m+l+1-n}{m\left(m-1\right)+2l+2} \to \frac{1}{m}, \text{ when } m \to \infty$$

and series diverge exponentially unless

$$n = m + l + 1$$

that is unless n is integer such that $n \geq l+1$. The energies are

$$E = -\frac{1}{2n^2}$$

2. (5 points)

$$x^2 y' + xy \ln y = y$$

Solution

$$x\left(x\frac{y'}{y} + \ln y\right) = 1$$

$$\left(x \ln y\right)' = x^{-1}$$

$$x \ln y = \ln \frac{x}{C}$$

$$y = \left(\frac{x}{C}\right)^{1/x}$$

3. (5 points)

$$y^{'} x - y = \sqrt{y^2 + x^2}$$

Solution

$$y = v\left(x\right) x$$

$$\left(v^{'} x + v\right) x - vx = x\sqrt{v^2+1}$$

$$v^{'} x = \sqrt{v^2+1}$$

$$\sinh^{-1} v = \ln\left(\frac{x}{a}\right)$$

$$v = \sinh\left(\ln \frac{x}{a}\right) = \left(\frac{x}{a} - \frac{a}{x}\right)/2$$

$$y = x\left(\frac{x}{a} - \frac{a}{x}\right)/2 = \frac{x^2}{2a} - \frac{a}{2}$$