MathPhys - Winter 2003
Final

1. Find the retarded Green's function of the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) G(x, x'; t) = \delta\left(x - x'\right) \delta\left(t\right)$$

for a string of length l whose ends are fixed.

Solution

Using

$$\delta\left(x - x'\right) = \frac{2}{l} \sum_n \sin \frac{n\pi x}{l} \sin \frac{n\pi x'}{l}$$

and

$$G(x, x'; t) = \sum_n G_n \sin \frac{n\pi x}{l} \sin \frac{n\pi x'}{l}$$

we find

$$\left(\frac{\partial^2}{\partial t^2} + \omega_n^2\right) G_n = \frac{2}{l} \delta\left(t\right)$$
$$\omega_n = \frac{cn\pi}{l}$$

whereof

$$G_n = \frac{2}{l} \frac{\sin \omega_n t}{\omega_n} \theta\left(t\right)$$

and finally

$$G(x, x'; t) = \theta\left(t\right) \frac{2}{l} \sum_n \frac{\sin \omega_n t}{\omega_n} \sin \frac{n\pi x}{l} \sin \frac{n\pi x'}{l}$$

2. Find the distribution of temperature in a rod $0 \leq x \leq l$ thermally insulated along the surface,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\kappa} \frac{\partial}{\partial t}\right) T = 0$$

if the temperature of its ends is maintained equal to zero and the initial temperature is T_0.

Solution

Using

$$T = \sum_n T_n \sin \frac{n\pi x}{l}$$

we find

$$\left[\left(\frac{n\pi}{l}\right)^2 + \frac{1}{\kappa} \frac{\partial}{\partial t}\right] T_n = 0$$

whereof

$$T_n = A_n \exp\left[-\kappa\left(\frac{n\pi}{l}\right)^2 t\right]$$

Consequently,

$$T = \sum_n A_n \exp\left[-\kappa\left(\frac{n\pi}{l}\right)^2 t\right] \sin\frac{n\pi x}{l}$$

and using initial condition

$$T_0 = \sum_n A_n \sin\frac{n\pi x}{l}$$

we finally find

$$A_n = \frac{4}{n\pi} T_0$$

when n is odd, so that

$$T = \frac{4T_0}{\pi} \sum_{n:\text{odd}} \frac{1}{n} \exp\left[-\kappa\left(\frac{n\pi}{l}\right)^2 t\right] \sin\frac{n\pi x}{l}$$

3. Find the electrostatic field inside a region, bounded by the conducting plates $y = 0$, $y = b$ and $x = 0$, if the plate $x = 0$ is charged to a potential V, the plates $y = 0$, $y = b$ are grounded (potential is zero), and if there is no charge inside the region

$$\nabla^2 V = 0$$

Solution

Expanding

$$V(x, y) = \sum_n V_n(x) \sin\frac{n\pi y}{b}$$

we find

$$\frac{d^2 V_n(x)}{dx^2} - \left(\frac{n\pi}{b}\right)^2 = 0$$

subject to the b.c.

$$V_n(0) = V$$

Consequently,

$$V_n(x) = V_n \exp\left(-\frac{n\pi x}{b}\right)$$

with

$$V = \sum_n V_n \sin\frac{n\pi y}{b}$$

that is

$$V_n = \frac{4V}{\pi}$$

when n is odd. Finally,

$$V(x, y) = \frac{4V}{\pi} \sum_{n:\text{odd}} \frac{1}{n} \exp\left(-\frac{n\pi x}{b}\right) \sin\frac{n\pi y}{b}$$

4. Find the temperature of the sphere of radius R the surface of which is maintained at T_0. At the initial time the temperature of the sphere was equal to zero.

Hint: Due to radial symmetry, the equation for the temperature is given by

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) T = 0$$

where the following change of variable

$$v = r\,(T - T_0)$$

might be useful.

Solution

$$T = T_0 + \Delta T$$

where

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) \Delta T = 0$$

Substituting

$$\Delta T = \frac{v}{r}$$

we find

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\frac{\partial}{\partial r} + \frac{2}{r^3} + \frac{2}{r^2}\frac{\partial}{\partial r} - \frac{2}{r^3} - \frac{1}{\kappa r}\frac{\partial}{\partial t}\right) v = 0$$

or

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) v = 0$$

with the b.c.

$$v\,(R) = 0,\ v\,(0) = 0$$

Consequently,

$$v\,(r,t) = \sum_n v_n\,(t) \sin\frac{n\pi r}{R}$$

so that

$$\left[\left(\frac{n\pi}{R}\right)^2 + \frac{1}{\kappa}\frac{\partial}{\partial t}\right] v = 0$$

and

$$v_n\,(t) = B_n \exp\left[-\kappa\left(\frac{n\pi}{R}\right)^2 t\right]$$

Then

$$T = T_0 + \sum_n B_n \exp\left[-\kappa\left(\frac{n\pi}{R}\right)^2 t\right] \frac{\sin\left(\frac{n\pi r}{R}\right)}{r}$$

Finally, using initial condition

$$0 = T_0 + \sum_n B_n \frac{\sin\left(\frac{n\pi r}{R}\right)}{r}$$

we find B_n (multiplying by $\frac{1}{r}\sin\left(\frac{m\pi r}{R}\right)$ and integrating with $4\pi\int_o^R drr^2$).