MathPhys - Fall 2002
Final

1. (20 points) Determine the forced oscillations of a system under a force

$$F(t) = F_0 \exp(-\alpha t)$$

if at time $t = 0$ the system is at rest in equilibrium ($x = \dot{x} = 0$). Solve the problem by two different techniques:

(a) (10 points) finding the complimentary function and the particular integral

(b) (10 points) using the Laplace transform

Solution

(a) Complimentary function

$$x_c = A \sin \omega t + B \cos \omega t$$

Look for a particular integral as

$$x_p = C \exp(-\alpha t)$$

Then

$$\left(\alpha^2 + \omega^2\right) C = \frac{F_0}{m}$$

and

$$x = x_c + x_p = A \sin \omega t + B \cos \omega t + \frac{F_0 \exp(-\alpha t)}{m\left(\alpha^2 + \omega^2\right)}$$

Using initial conditions

$$\begin{aligned} B + \frac{F_0}{m\left(\alpha^2 + \omega^2\right)} &= 0 \\ A\omega - \alpha \frac{F_0}{m\left(\alpha^2 + \omega^2\right)} &= 0 \end{aligned}$$

and

$$\begin{aligned} x &= \frac{\alpha}{\omega} \frac{F_0}{m\left(\alpha^2 + \omega^2\right)} \sin \omega t - \frac{F_0}{m\left(\alpha^2 + \omega^2\right)} \cos \omega t + \frac{F_0 \exp(-\alpha t)}{m\left(\alpha^2 + \omega^2\right)} \\ &= \frac{F_0}{m\left(\alpha^2 + \omega^2\right)} \left(\exp(-\alpha t) - \cos \omega t + \frac{\alpha}{\omega} \sin \omega t\right) \end{aligned}$$

(b) Taking the Laplace transform of the equation of motion, $X(s) = \mathcal{L}[x(t)]$,

$$\left(s^2 + \omega^2\right) X = \frac{F_0}{m(\alpha + s)}$$

whereof the Laplace inversion formula

$$\begin{aligned} x &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F_0 \exp(st)}{m(\alpha + s)\left(s^2 + \omega^2\right)} = \frac{F_0}{m} \left(\frac{\exp(-\alpha t)}{\left(\alpha^2 + \omega^2\right)} + \frac{1}{2i\omega} \left(\frac{\exp(i\omega t)}{(i\omega + \alpha)} - \frac{\exp(-i\omega t)}{(-i\omega + \alpha)} \right) \right) \\ &= \frac{F_0}{m} \left(\frac{\exp(-\alpha t)}{\left(\alpha^2 + \omega^2\right)} + \frac{\cos \omega t}{\left(\alpha^2 + \omega^2\right)} + \frac{\alpha}{\omega} \frac{\sin \omega t}{\left(\alpha^2 + \omega^2\right)} \right) = \frac{F_0}{m\left(\alpha^2 + \omega^2\right)} \left(\exp(-\alpha t) - \cos \omega t + \frac{\alpha}{\omega} \sin \omega t\right) \end{aligned}$$

2. (20 points) Fourier and Laplace transform of the product

(a) (10 points) Suppose

$$\begin{aligned} \mathcal{F}\left[f_1(x)\right] &= g_1(y) \\ \mathcal{F}\left[f_2(x)\right] &= g_2(y) \end{aligned}$$

Express

$$\mathcal{F}\left[f_1 f_2\right]$$

in terms of g_1 and g_2.

Solution

$$\begin{aligned} \mathcal{F}\left[f_1 f_2\right] &= \int_{-\infty}^{\infty} f_1(x) f_2(x) \exp(ixy)\, dx \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(y') \exp(-ixy')\, dy'\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g_2(y'') \exp(-ixy'')\, dy''\right) \exp(ixy)\, dx \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy' dy'' g_1(y') g_2(y'') \int_{-\infty}^{\infty} \exp(-ixy') \exp(-ixy'') \exp(ixy)\, dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy' dy'' g_1(y') g_2(y'') \delta(y - y' - y'') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy' g_1(y') g_2(y - y') \end{aligned}$$

(b) (10 points) Suppose

$$\begin{aligned} \mathcal{L}\left[f_1(x)\right] &= g_1(s) \\ \mathcal{L}\left[f_2(x)\right] &= g_2(s) \end{aligned}$$

Express

$$\mathcal{L}\left[f_1 f_2\right]$$

in terms of g_1 and g_2.

Solution

$$\begin{aligned} \mathcal{L}\left[f_1 f_2\right] &= \int_{-\infty}^{\infty} f_1(x) f_2(x) \exp(-xs)\, dx \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_1(s') \exp(xs')\, ds'\right) f_2(x) \exp(-xs)\, dx \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_1(s')\, ds' \int_{-\infty}^{\infty} f_2(x) \exp(-x(s - s'))\, dx \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_1(s') g_2(s - s')\, ds' \end{aligned}$$