Remarks

Problem 26

$v_{1,2}$ defined up to a const $c_{1,2}$
but C_1C_2 cancels by C_1C_2 due to
w in the denominator, so
G is uniquely defined

Table 9.1

A. Arfken: $(p(x)y')' + q(x)y = Ly$

E.v. problem $Ly + \lambda w y = 0$

$$-(p(x)y') - q(x)y = \lambda w y$$

$q \to -q$ in our notations

B. Legendre: $P_\ell(x)$ $\qquad \lambda = \ell(\ell+1)$

Orthogonality involves $P_\ell(x)$ and $P_{\ell'}(x)$

Assoc. Legendre: $P_\ell^m(x)$ $\qquad \lambda = \ell(\ell+1)$

Orthogonality involves $P_\ell^m(x)$ and $P_{\ell'}^m(x)$

Bessel: $J_n(ax)$

Orthogonality involves $J_m(ax)$ and $J_m(bx)$

Example

Find the Green's function of the boundary value problem

$$-u'' = f(x), \quad u(0) = u(1) = 0$$

and solve the problem for $f(x) = 1$

A. $v''_{1,2} = 0$, $\quad v_{1,2} = C'_{1,2} + C''_{1,2}x$

a) $v_1(0) = 0 \Rightarrow C'_1 = 0$, $\quad v_1 = x$

b) $v_2(1) = 0 \Rightarrow C'_2 + C'_1 = 0$, $\quad v_2 = x - 1$

$$w = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = x - (x-1) = 1, \quad p = 1$$

$$G(x,y) = -\frac{1}{wp}\begin{cases} v_1(x)v_2(y), & x \le y \\ v_1(y)v_2(x), & y \le x \end{cases} = \begin{cases} x(1-y) \\ y(1-x) \end{cases}$$

B. $$u = \int_0^1 G(x,y) f(y) = \int_0^x (1-x)y\,dy + \int_x^1 x(1-y)\,dy$$

$$= (1-x)\frac{x^2}{2} - x\frac{(1-y)^2}{2}\Big|_x^1 = (1-x)\frac{x^2}{2} + \frac{x}{2}\frac{(1-x)^2}{2}$$

$$= (1-x)\frac{x}{2}\left[x + 1 - x\right] = \frac{x}{2}(1-x)$$

Check $\quad -u'' = 1$, $\quad u' = -x + C_1$, $\quad u'' = -\frac{x^2}{2} + C_1x + C_2$

a) $u(0) = 0 \Rightarrow C_2 = 0$

b) $u(1) = 0 \Rightarrow -\frac{1}{2} + C_1 = 0$, $C_1 = \frac{1}{2}$

$$u = -\frac{x^2}{2} + \frac{x}{2} = \frac{x}{2}(1-x)$$