$$y'' + P(x)y' + Q(x)y = F(x)$$

$p(x)$: "integrating factor"

$$\underline{y''p + pPy'} + pQy = pF$$

$$y''p + pPy' = (y'p)' = y''p + y'p' \Rightarrow p' = pP$$

$$(py')' + \underbrace{pQ}_{-q}y = \underbrace{pF}_{-f}$$

$$(py')' - qy = -f$$

$$-(py')' + qy = f$$

- - - - - - - - - - - -

$L = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q$ is Hermitian

$$\int_a^b (Lu^*)\, v\, dx = \int_a^b q v u^* dx - \int_a^b v\left[\frac{d}{dx}\left(p\frac{d}{dx}\right)\right]u^* dx$$

$$= \int_a^b q v u^* dx - vp\frac{du^*}{dx}\Big|_a^b + \int_a^b dx\, p\frac{du^*}{dx}\frac{dv}{dx}$$

$$\alpha u^*(a) - A u^{*\prime}(a) = 0 \qquad \beta u^*(b) + B u^{*\prime}(b) = 0$$

$$= \int_a^b q v u^* dx + \frac{\alpha}{A} v(a) u^*(a) + \frac{\beta}{B} v(b) u^*(b) + \int_a^b dx\, p u^{*\prime} v'$$

$$\alpha v(a) - A v'(a) = 0 \qquad \beta v(b) + B v'(b) = 0$$

$$= \int_a^b q v u^* dx - u^* p\frac{dv}{dx}\Big|_a^b + \int_a^b dx\, p\frac{du^*}{dx}\frac{dv}{dx}$$

$$\int_a^b u^* L v = \int_a^b q u^* v \, dx - \int_a^b u^* \left[\frac{d}{dx}\left(p \frac{d}{dx}\right)\right] v \, dx$$

$$= \int_a^b q u^* v \, dx - u^* p \frac{dv}{dx}\Big|_a^b + \int_a^b p \frac{dv}{dx}\frac{du^*}{dx} dx$$

$$(Lu, v) = \int_a^b (Lu^*)\, v \, dx = \int_a^b u^* L v = (u, Lv)$$

1) $$(Lu, u) = \int_a^b dx \left[q|u|^2 + p\left|\frac{du}{dx}\right|^2 \right]$$

$$+ \frac{\alpha}{A} |u(a)|^2 + \frac{\beta}{B} |u(b)|^2 \geq 0$$

Energy integral

$$= \lambda(u,u) = \lambda \underbrace{\int dx\, |u|^2}_{=1 \text{ for normalized } u}$$

2) $Lu_1 = \lambda_1 u_1 \quad , \quad Lu_2 = \lambda_2 u_2$

$$(Lu_1, u_2) = \lambda_1^* (u_1, u_2) = \lambda_1^* \int u_1^* u_2 \, dx$$

$$\parallel$$

$$(u_1, Lu_2) = \lambda_2 (u_1, u_2) = \lambda_2 \int u_1^* u_2 \, dx$$

$$\lambda_1^*(u_1, u_2) - \lambda_2 (u_1, u_2) = 0 \Rightarrow (u_1, u_2) = 0$$

e.f. corresponding to different e.v. are orthogonal

3) e.f. can be chosen to be real

$$Lu = \lambda u, \quad u = \bar{u} + i\bar{\bar{u}} \Rightarrow L\bar{u} = \lambda \bar{u}, \; L\bar{\bar{u}} = \lambda \bar{\bar{u}}$$

4) Any function satisfying the b.c. can be expanded in terms of e.f. of L

Example

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \quad , \quad \psi(a) = \psi(b) = 0$$

$$V(x) = \infty, \ x \notin [a,b]$$

Generalization

$$-\vec{\nabla}\cdot\left(p(\vec{r})\vec{\nabla}u(\vec{r})\right) + q(\vec{r})u(\vec{r}) = \lambda u(\vec{r})$$

$$\left(\alpha(\vec{r})u(\vec{r}) + \beta(\vec{r})\frac{\partial u}{\partial \vec{n}}\right)\Big|_{\vec{r}\in S} = 0$$

$$\alpha \geq 0, \ \beta \geq 0, \ \left(\alpha(\vec{r}) + \beta(\vec{r})\right)\Big|_{\vec{r}\in S} > 0$$

Self-adjoint operator

$$\mathcal{L}u = p_0 u''(x) + p_1 u'(x) + p_2 u(x)$$

$$\bar{\mathcal{L}}u = (p_0 u)'' - (p_1 u)' + p_2 u$$

$$\mathcal{L}u = \bar{\mathcal{L}}u \Rightarrow \begin{cases} 2p_0' - p_1 = p_1 \\ p_0'' - p_1' + p_2 = p_2 \end{cases} \Rightarrow p_0' = p_1$$

Hermitian: self-adjoint + b.c.

Weighting function

$$\mathcal{L}u(x) = \lambda w(x) u(x)$$

$$\int |u_i|^2 w(x)\,dx = 1$$ normalization

$$\int u_i(x) u_j^*(x) w(x)\,dx = 0$$ orthogonality

TABLE 9.1

Equation	$p(x)$	$q(x)$	λ	$w(x)$
Legendre	$1 - x^2$	0	$l(l+1)$	1
Shifted Legendre	$x(1-x)$	0	$l(l+1)$	1
Associated Legendre	$1 - x^2$	$-m^2/(1-x^2)$	$l(l+1)$	1
Chebyshev I	$(1-x^2)^{1/2}$	0	n^2	$(1-x^2)^{-1/2}$
Shifted Chebyshev I	$[x(1-x)]^{1/2}$	0	n^2	$[x(1-x)]^{-1/2}$
Chebyshev II	$(1-x^2)^{3/2}$	0	$n(n+2)$	$(1-x^2)^{1/2}$
Ultraspherical (Gegenbauer)	$(1-x^2)^{\alpha+1/2}$	0	$n(n+2\alpha)$	$(1-x^2)^{\alpha-1/2}$
Bessel*	x	$-\dfrac{n^2}{x}$	a^2	x
Laguerre	xe^{-x}	0	α	e^{-x}
Associated Laguerre	$x^{k+1}e^{-x}$	0	$\alpha - k$	$x^k e^{-x}$
Hermite	e^{-x^2}	0	2α	e^{-x^2}
Simple harmonic oscillator†	1	0	n^2	1

*Orthogonality of Bessel functions is rather special. Compare Section 11.2 for details. A second type of orthogonality is developed in Section 11.7.

†This will form the basis for Chapter 14, Fourier series.