Generalized functions

Example: density of a point mass 1 at origin

$$f_\varepsilon(x) = \begin{cases} \frac{3}{4\pi\varepsilon^3}, & |x| < \varepsilon \\ 0, & |x| > \varepsilon \end{cases} \xrightarrow{\varepsilon \to 0} \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Must have $\int_V \delta(x)\,dx = \begin{cases} 1, & 0 \in V \\ 0, & 0 \notin V \end{cases}$

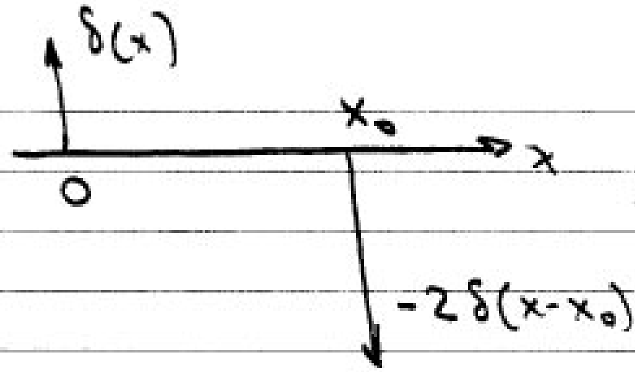
"weak limit" $\quad f_\varepsilon(x) \xrightarrow{\varepsilon \to 0} \delta(x)$

$$\lim_{\varepsilon \to 0} \int f_\varepsilon(x)\,\phi(x)\,dx = \phi(0) \equiv (\delta, \phi)$$

Exercise (1D)

Show that

$$f_\varepsilon(x) = \begin{cases} \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}} \\ \frac{1}{\pi x} \sin\frac{x}{\varepsilon} \\ \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} \\ \frac{\varepsilon}{\pi x^2} \sin^2\frac{x}{\varepsilon} \end{cases} \xrightarrow{\varepsilon \to 0} \delta(x)$$



Example

Simple (single) layer on surface S
density μ

$$(\mu\delta_S, \varphi) = \int_S \mu(x)\,\varphi(x)\,dS$$

Example

Show that

$$\frac{1}{x+i\varepsilon} \xrightarrow{\varepsilon\to 0} -i\pi\,\delta(x) + \mathcal{P}\,\frac{1}{x}$$

That is

$$\int \frac{\varphi(x)}{x+i\varepsilon}\,dx \xrightarrow{\varepsilon\to 0} -i\pi\,\varphi(0) + \mathcal{P}\int \frac{\varphi(x)}{x}\,dx$$

$$\lim_{\varepsilon\to 0}\int \frac{\phi(x)}{x+i\varepsilon}\,dx = \phi(0)\lim_{\varepsilon\to 0}\int \frac{x-i\varepsilon}{x^2+\varepsilon^2}\,dx$$

$$+\lim_{\varepsilon\to 0}\int \frac{x-i\varepsilon}{x^2+\varepsilon^2}\left[\phi(x)-\phi(0)\right]dx$$

$$\int \overbrace{\frac{x}{x^2+\varepsilon^2}}^{\text{odd fn}}\,dx = 0$$

$$\int \frac{-i\varepsilon}{x^2+\varepsilon^2}\,dx = -i\pi$$

$$\lim_{\varepsilon\to 0}\int \frac{x-i\varepsilon}{x^2+\varepsilon^2}\left[\phi(x)-\phi(0)\right]$$

$$= \int \frac{\phi(x)-\phi(0)}{x} = \mathcal{P}\int \frac{\phi(x)}{x}$$

$$\left(\text{if } \mathcal{P}, \text{ then } \int \frac{\phi(0)}{x} = 0\right)$$

So $$\boxed{\frac{1}{x\pm i0} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x}}$$

Alternative proof

$$\int \frac{\phi(x)}{x+i\varepsilon} = \int_C \frac{\phi(z)}{z+i\varepsilon}$$

$$= \mathcal{P}\int \frac{\phi(x)}{x} + \underbrace{\int_{\frown} \frac{\phi(z)}{z}\,dz}$$

$$= \lim_{\rho\to 0}\int_{\pi}^{0} \frac{\phi(\rho e^{i\phi})}{\cancel{\rho e^{i\phi}}}\,\cancel{\rho e^{i\phi}}\, i\, d\phi = -i\pi\phi(0)$$

Exercise

Show that

$$\lim_{t\to\infty} \frac{e^{\pm ixt}}{x-i0} = \begin{cases} 2\pi i\,\delta(x) \\ 0 \end{cases}$$

$$\lim_{t\to\infty} \frac{e^{\pm ixt}}{x+i0} = \begin{cases} 0 \\ -2\pi i\,\delta(x) \end{cases}$$

Derivadive

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x_1, x_2, \dots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

$$D = (D_1, D_2, \dots, D_n), \quad D_j = \frac{\partial}{\partial x_j}$$

other notations

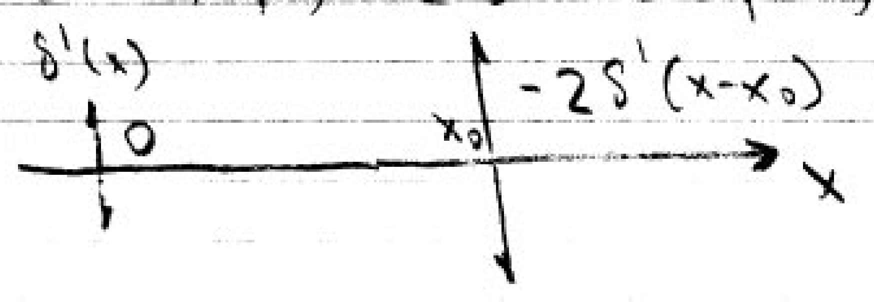
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad \alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$$

Definition of derivadive of generalized fn

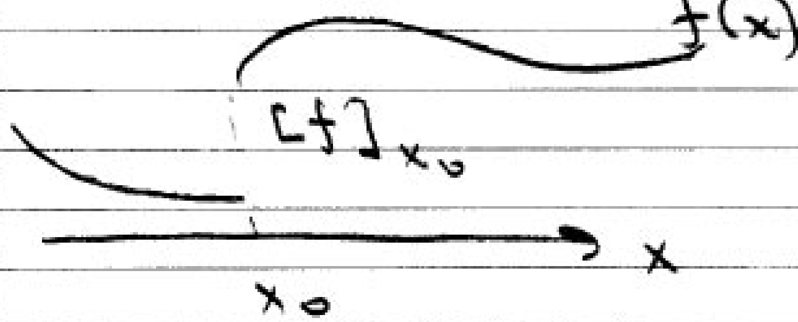
$$(D^\alpha f, \phi) = (-1)^{|\alpha|} (f, D^\alpha \phi)$$

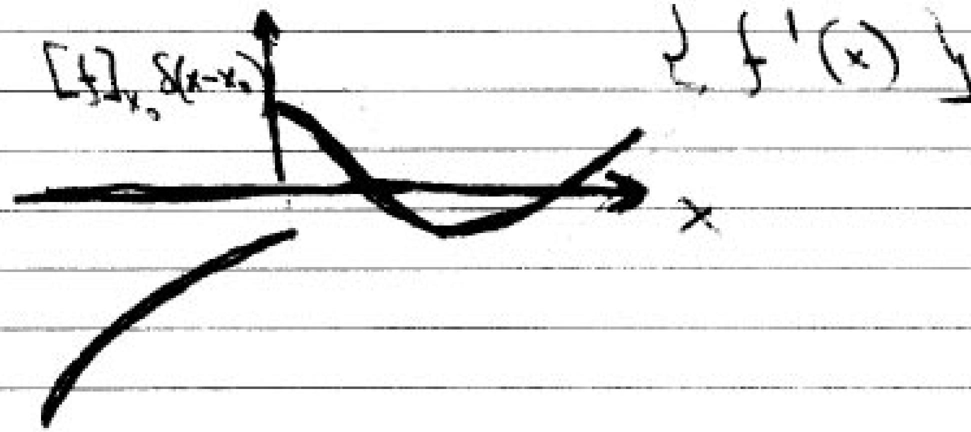
In particular

$$(D^\alpha \delta, \phi) = (-1)^{|\alpha|} D^\alpha \phi(0)$$



Example





$$[f]_{x_0} = f(x_0+0) - f(x_0-0)$$

$$f'(x) = \{f'(x)\} + [f]_{x_0}\delta(x-x_0)$$

$$(f', \phi) = -(f, \phi') = -\int f(x)\phi'(x)\,dx$$

$$= [f(x_0+0) - f(x_0-0)]\,\phi(x_0) + \int \{f'(x)\}\,\phi(x)\,dx$$

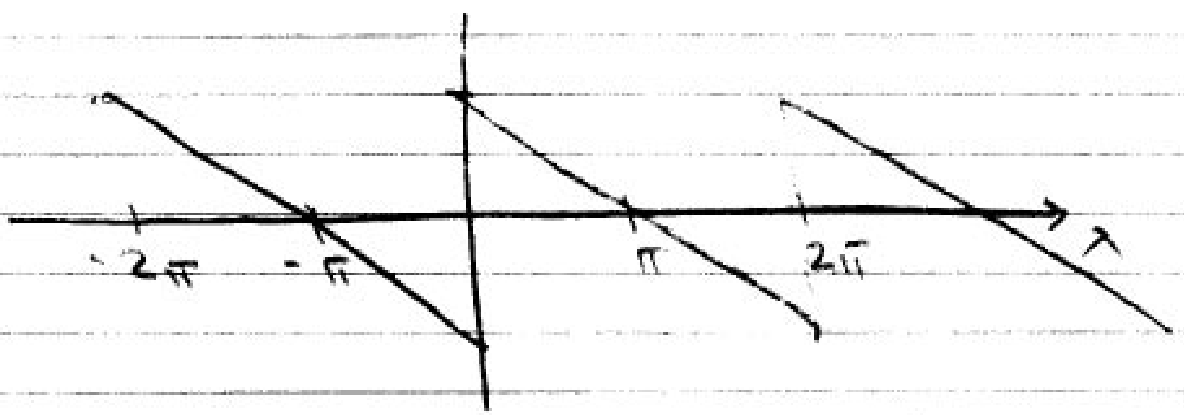
integrating by parts

In particular, for step-function

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$\theta'(x) = \delta(x)$$

<u>Example</u>



$$f_0(x) = \frac{1}{2} - \frac{x}{2\pi}, \quad x \in [0, 2\pi)$$

$$f_0' = -\frac{1}{2\pi} + \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$$

Consider $\displaystyle\int_0^x f_0(x')\,dx' = \frac{x}{2} - \frac{x^2}{4\pi}, \; x \in [0, 2\pi)$

FT $\displaystyle = \frac{\pi}{6} - \frac{1}{2\pi} \sum_{k \neq 0} \frac{1}{k^2} e^{ikx}$

$$f_0' = -\frac{1}{2\pi} + \sum_k \delta(x - 2k\pi) = \frac{1}{2\pi} \sum_{k \neq 0} e^{i\cdots}$$

$$\boxed{\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx} = \sum_{k=-\infty}^{\infty}\delta(x-2k\pi)}$$

Poisson formula

<u>Example</u>

$$\frac{\partial f}{\partial x_i} = \left\{\frac{\partial f}{\partial x_i}\right\} + [f]_S \cos(\hat{n}x_i)\,\delta_S$$

$$\nabla^2 f = \{\nabla^2 f\} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left([f]_S \cos(\hat{n}x_i)\,\delta_S\right)$$

$$+ \sum_{i=1}^{n} \left[\left\{\frac{\partial f}{\partial x_i}\right\}\right]_S \cos(\hat{n}x_i)\,\delta_S$$

$$= \{\nabla^2 f\} + \left[\frac{\partial f}{\partial \vec{n}}\right]_S \delta_S + \frac{\partial}{\partial \vec{n}}\left([f]_S\,\delta_S\right)$$

f=0 outside

$$= \{\nabla^2 f\} - \frac{\partial f}{\partial \vec{n}}\delta_S - \frac{\partial}{\partial \vec{n}}(f\delta_S)$$

applying to ϕ

$$\int_G \left(f\nabla^2\phi - \phi\nabla^2 f\right) = \int_S \left(f\frac{\partial \phi}{\partial n} - \phi\frac{\partial f}{\partial n}\right) dS$$

Example

Show that $G(t) = \theta(t) z(t)$

where $Lz = z^{(n)} + a_1(t) z^{(n-1)} + \dots a_n(t) z = 0$

and $z(0) = z'(0) = \dots = z^{(n-2)}(0) = 0$

$z^{(n-1)}(0) = 1$

satisfies $LG(t) = \delta(t)$

$$G' = \theta(t) z'(t) + \overbrace{\delta(t) z(t)}^{\propto\, z(0) = 0}$$

$\vdots$

$$G^{(n-1)} = \theta(t) z^{(n-1)}(t)$$

$$G^{(n)} = \delta(t) + \theta(t) z^{(n)}(t)$$

So $LG = \theta(t) Lz + \delta(t) = \delta(t)$

For instance

$G(t) = \theta(t) e^{-at}$, $L = \frac{d}{dt} + a$

$G(t) = \theta(t) \frac{\sin at}{a}$, $L = \frac{d^2}{dt^2} + a^2$

Retarded GF of diffusion Eq.

$$\frac{\partial G}{\partial t} - D\nabla^2 G = \delta(\vec{r}, t) \longleftarrow FT$$

$$\frac{\partial G}{\partial t} + Dq^2 G = \delta(t)$$

$$G = \Theta(t) e^{-Dq^2 t}$$

$$G(\vec{r}, t) = \Theta(t) \int d\vec{q}\, e^{i\vec{q}\cdot\vec{r} - Dq^2 t}$$

$$= \Theta(t) \prod_{i=1}^{d} \int dq_i\, e^{iq_i x_i - Dq_i^2 t}$$

$$= \frac{\Theta(t)}{(2\sqrt{\pi D t})^d} e^{-\frac{r^2}{4Dt}}$$

$$\frac{\partial^2 G}{\partial t^2} - c^2 \nabla^2 G = \delta(\vec{r}, t)$$

$$\delta(\vec{r}) = \frac{1}{(2\pi)^d} \int e^{i\vec{k}\cdot\vec{r}} \, d\vec{r}$$

$$G(\vec{r}, t) = \frac{1}{(2\pi)^d} \int e^{i\vec{k}\cdot\vec{r}} \, G(\vec{k}, t) \, d\vec{k}$$

$$\frac{\partial^2 G}{\partial t^2} + c^2 k^2 G = \delta(t)$$

$$G(\vec{k}, t) = \frac{\sin ckt}{ck} \Theta(t) \qquad (*)$$

Consider 3D, S_R: sphere of radius R

$$\int_{S_R} e^{-i\vec{k}\cdot\vec{r}} \, ds = R^2 \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \, e^{-ikR\cos\theta}$$

$$= 4\pi R \frac{\sin kR}{k} \qquad (**)$$

$$G(\vec{k}, t) = \int G(\vec{r}, t) e^{-i\vec{k}\cdot\vec{r}} \, d\vec{r} \qquad (***)$$

From $(*) - (***)$

$$\boxed{G_3(\vec{r}, t) = \frac{1}{4\pi c^2 t} \delta_{S_{ct}}(\vec{r})} \qquad \begin{pmatrix} \text{Notice:} \\ R = ct \end{pmatrix}$$

This can be also written as

$$G_3(\vec{r},t) = \frac{1}{2\pi c}\,\delta(c^2t^2 - r^2)$$

$$= \frac{1}{4\pi r c}\,\delta(ct - r)$$

$$= \frac{1}{4\pi c^2 t}\,\delta(ct - r)$$

$$(\nabla^2+k^2)G(\vec{r})=\delta(\vec{r})$$

$$G(\vec{r})=\frac{1}{(2\pi)^d}\int G(\vec{q})e^{i\vec{q}\cdot\vec{r}}\,d\vec{q}$$

$$\delta(\vec{r})=\frac{1}{(2\pi)^d}\int e^{i\vec{q}\cdot\vec{r}}\,d\vec{q}$$

$$G(\vec{q})=\frac{1}{k^2-q^2}$$

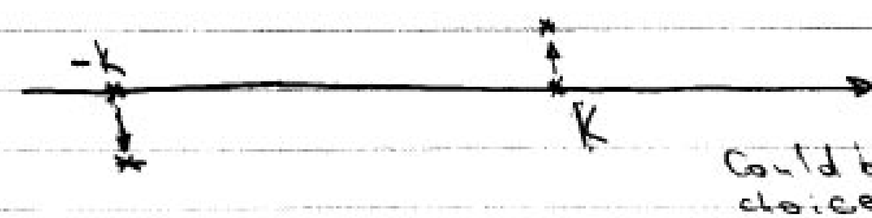
$$G_3(\vec{r})=\frac{1}{(2\pi)^3}\int\frac{e^{i\vec{q}\cdot\vec{r}}}{k^2-q^2}\,d\vec{q}$$

$$=\frac{1}{(2\pi)^2}\int_0^\infty dq\,q^2\int_0^\pi d\theta\,\sin\theta\,\frac{e^{iqr\cos\theta}}{k^2-q^2}$$

$$=\frac{1}{(2\pi)^2}\int_0^\infty dq\,\frac{q^2}{k^2-q^2}\int_{-1}^{1}dx\,e^{iqrx}$$

$$=\frac{1}{2\pi^2 r}\int_0^\infty dq\,\frac{q\sin qr}{k^2-q^2}=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dq\,\frac{q\sin qr}{k^2-q^2}$$

$$=\frac{1}{(2\pi)^2 r}\frac{1}{2i}\int_{-\infty}^{\infty}dq\,\frac{q\left(e^{iqr}-e^{-iqr}\right)}{k^2-q^2}$$

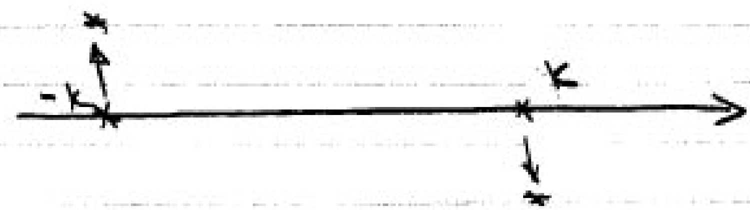


Could be other!! choices

(*)

$$= -\frac{1}{(2\pi)^2 r}\,\frac{2\pi i}{2i}\left\{ \mathrm{Res}\left(\frac{q e^{iqr}}{q^2-k^2}, k\right) + \mathrm{Res}\left(\frac{q e^{-iqr}}{q^2-k^2}, -k\right) \right\}$$

$$= -\frac{1}{4\pi r}\left\{\frac{k e^{ikr}}{2k} + \frac{(-k)e^{ikr}}{(-2k)}\right\} = \underline{\underline{-\frac{1}{4\pi r}e^{ikr}}}$$



$$\underline{\underline{-\frac{1}{4\pi r}e^{-ikr}}}$$

(*)