Bessel function

A. $x \to 0$ $\quad x^2 y'' + x y' - m^2 y = 0$, $\quad y = x^S$

$$S(S-1) + S - m^2 = 0, \quad S = \pm m$$

B. $y = x^{\pm m} u$

$$x u'' + (1 \pm 2m) u' + x u = 0, \quad u = \sum_{n=0}^{\infty} c_n x^n$$

$$\left[n(n+2) + (n+2)(1 \pm 2m)\right] c_{n+2} + c_n = 0$$

$$\left[(n+2 \pm 2m)(n+2)\right] c_{n+2} = c_n, \quad c_{n+2} = c_n \frac{-1}{(n+2)(n+2 \pm 2m)}$$

$$\sum_{n=0} n(n-1) c_n x^{n-1} + \sum_{n=0} (1 \pm 2m) n c_n x^{n-1} + \sum_{n=0} c_n x^{n+1} = 0$$

$$\sum_{n=0} n(n \pm 2m) c_n x^{n-1} + \sum_{n=0} c_n x^{n+1} = 0$$

$$\sum_{n=2} n(n \pm 2m) c_n x^{n-1} + \sum_{n=0} c_n x^{n+1} + \cancel{c_1 (1 \pm 2m)} = 0$$

$n-2 \to n$

$\div\ c_1(1 \pm 2m) = 0 \qquad c_1 = 0$, or $m = \mp \frac{1}{2}$

a) $m = -\frac{1}{2}$, then $c_1(1+2m) = 0$ leads to $c_1 = 0$ (and $c_{2k+1} = 0$)

for $y_+ = x^{1/2} \sum_{n=0} c_n^{(+)} x^n$. But $c_n^{(+)} = c_{n+1}^{(-)}$

analogously

b) $m = \frac{1}{2}$ for $y_- = x^{-1/2} \sum_{n=0} c_n^{(-)} x^n$ part of y_- with $c_{2k+1}^{(-)}$ coincides with $c^{(+)}$

C. m: integer > 0 (≥ 1) *

$$C_n^{(-)} = -C_{n+2}^{(-)} (n+2)(n+2-2m)$$

$$C^{(-)}_{n \leq 2m-2} = 0 \qquad y_- = x^{-m} \sum_{\substack{n \geq 2m \\ n:\text{even}}}^{\infty} C_n^{(-)} x^n$$

$$y_- = x^m \sum_{\substack{n-2m \geq 0 \\ n:\text{even}}}^{\infty} C_n^{(-)} x^{n-2m} = x^m \sum_{\substack{k \geq 0 \\ k:\text{even}}} C_{k+2m}^{(-)} x^k$$

$$y_+ \qquad = x^m \sum_{\substack{k \geq 0 \\ k:\text{even}}} C_k^{(+)} x^k$$

$$\frac{C^{(-)}_{k+2+2m}}{C^{(-)}_{k+2m}} = \frac{-1}{(k+2+2m)(k+2+2m-2m)} = \frac{1}{(k+2)(k+2+2m)}$$

$$= \frac{C^{(+)}_{k+2}}{C^{(+)}_{k}}$$

$$\frac{y_-}{y_+} = \text{const} \; !$$

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* $m=0$

$$Ly(x,s) = s^2 x^s \qquad \frac{\partial}{\partial s} Ly(x,s)\Big|_{s=0} = 2s x^s + s^2 \ln x \, x^s \Big|_{s=0} = 0$$

$$\bar{y} = \frac{\partial}{\partial s} y(x,s)\Big|_{s=0} : \text{second solution} \leftarrow L\left(\frac{\partial}{\partial s} y(x,s)\Big|_{s=0}\right) = 0$$