MathPhys 15-Phys-721
Winter 2002 Midterm
Wednesday, February 6

1. (9 points) Solve the differential equation

$$y'' + y = \cos t$$

given the initial conditions

$$y(0) = y'(0) = 0$$

You **_must_** use the Laplace transform to solve the problem - other methods will not be accepted.

Solution

Denoting

$$Y(p) = \mathcal{L}[y(t)]$$

we find

$$\left(p^2+1\right)Y(p) = \operatorname{Re}\left(\frac{1}{p-i}\right) = \frac{p}{p^2+1}$$

or

$$Y(p) = \frac{p}{\left(p^2+1\right)^2} = \frac{1}{4i}\left[\frac{1}{(p-i)^2} - \frac{1}{(p+i)^2}\right]$$

The Laplace inversion integral gives

$$y(t) = \frac{1}{4i}\left[\frac{\partial \exp(pt)}{\partial p}\,|_{p=i} - \frac{\partial \exp(pt)}{\partial p}\,|_{p=-i}\right] = \frac{1}{4i}\left[t\exp(it) - t\exp(-it)\right] = \frac{t\sin t}{2}$$

2. (6 points) Find the Fourier transform of the function

$$f(x) = \begin{matrix} 1 & |x|<1 \\ \frac{1}{2} & |x|=1 \\ 0 & |x|>1 \end{matrix}$$

Solution

Since $f(x)$ is an even function of x,

$$\begin{aligned} g(y) &= \mathcal{F}[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos(xy)\,dx \\ &= \frac{2}{\pi}\int_0^1 \cos(xy)\,dx = \frac{2}{\pi}\frac{\sin(y)}{y} \end{aligned}$$

3. (15 points) Expand the function

$$f(x) = \begin{matrix} 0 & x\in[-1,0) \\ 1 & x\in(0,1] \end{matrix}$$

in a series of Legendre polynomials

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n P_n(x) \\ c_n &= \frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx \end{aligned}$$

Hint: One possible way to solve the problem is as follows: Use the generating function

$$F(h,x) = \frac{1}{\sqrt{1-2hx+h^2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

to derive the relationship

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)\,P_n(x)$$

and also to evaluate $P_n(0)$.

Solution

Differentiating F on h and on x, we find

$$(n+1) P_{n+1}(x) - (2n+1) x P_n(x) + n P_{n-1}(x) = 0$$

and

$$P'_n(x) - 2x P'_{n-1}(x) + P'_{n-2}(x) = P_{n-1}(x)$$

respectively. Combining the differentiation on x of the first equation with the shift $n \to n+1$ in the second equation leads, upon elimination of $P'_{n-1}(x)$, to

$$P'_{n+1} - x P'_n - (n+1) P_n = 0$$

Analogously, we find

$$x P'_n - P'_{n-1} - n P_n = 0$$

and

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x)$$

Consequently,

$$c_n = \frac{2n+1}{2} \int_0^1 P_n(x)\, dx = \frac{1}{2} \int_0^1 \left[P'_{n+1}(x) - P'_{n-1}(x) \right] dx = \frac{1}{2} \left[-P_{n+1}(0) + P_{n-1}(0) \right]$$

But, using the generating function at $x = 0$, one finds that

$$P_n(0) = \frac{(n-1)!!\, (-1)^{n/2}}{2^{n/2} (n/2)!}$$

when n is even and 0 otherwise. Denoting $n = 2k+1$,

$$\begin{aligned} c_{2k+1} &= \frac{1}{2} \left[-P_{2k+2}(0) + P_{2k}(0) \right] = \frac{1}{2} \left[-\frac{(2k+1)!!\, (-1)^{k+1}}{2^{k+1} (k+1)!} + \frac{(2k-1)!!\, (-1)^k}{2^k k!} \right] \\ &= \frac{(2k-1)!!\, (-1)^k}{22^k k!} \left[\frac{(2k+1)}{2(k+1)} + 1 \right] = \frac{(2k-1)!!\, (-1)^k}{2^{k+2} k!} \frac{4k+3}{k+1} \end{aligned}$$

4. (10 points) The generating function for the Bessel functions is

$$F(h, x) = \exp\left[\frac{x}{2} \left(h - \frac{1}{h} \right) \right] = \sum_{n=-\infty}^{\infty} h^n J_n(x)$$

At large x, the asymptotic behavior of $J_0(x)$ is

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} \right)$$

Using the generating function, find the relationship between $J_0(x)$ and $J_1(x)$ and the asymptotic behavior of the latter.

Solution

Differentiating $F(h, x)$ on h, we find

$$\frac{x}{2}(1 + \frac{1}{h^2}) F(h, x) = \sum_{n=-\infty}^{\infty} n h^{n-1} J_n(x)$$

whereof

$$\frac{x}{2} (J_{n+1} + J_{n-1}) = n J_n$$

and

$$J_{n+1} + J_{n-1} = \frac{2n}{x} J_n$$

Differentiating $F(h,x)$ on x, we find

$$\frac{1}{2}\left(h-\frac{1}{h}\right)F(h,x)=\sum_{n=-\infty}^{\infty}h^nJ_n'(x)$$

whereof

$$\frac{1}{2}(-J_{n+1}+J_{n-1})=J_n'$$

and

$$J_{n+1}-J_{n-1}=-2J_n'$$

Consequently,

$$J_{n+1}=\frac{n}{x}J_n-J_n'$$

For $n=0$

$$J_1=-J_0'\approx\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\pi}{4}\right)=-\sqrt{\frac{2}{\pi x}}\cos\left(x+\frac{\pi}{4}\right)$$