MathPhys 15-Phys-721
Fall 2001 Midterm
Tuesday, November 6

1. (12 points) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x}$$

Solution

General solution of the homogeneous equation, $y \propto e^{mx}$

$$m^2 - 4m + 4 = (m-2)^2 = 0$$

that is

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

Look for the particular integral as $y = Ax^2 e^{2x}$ and find

$$2Ae^{2x} = 2e^{2x}$$

General solution of the inhomogeneous equation

$$y = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x}$$

2. (12 points) Find the sum

$$\sum_{k=1}^{\infty} \frac{1}{k(2k+1)}$$

Solution

Consider

$$S(x) = \sum_{k=1}^{\infty} \frac{x^{2k+1}}{k(2k+1)}$$

and

$$\frac{dS(x)}{dx} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} = -\ln\left(1-x^2\right)$$

whereof

$$\begin{aligned} S(x) &= -\int dx \ln\left(1-x^2\right) = \int d(1-x)\ln(1-x) - \int d(1+x)\ln(1+x) \\ &= (1-x)\left[\ln(1-x)-1\right] - (1+x)\left[\ln(1+x)-1\right] + C \end{aligned}$$

Find $C=0$ from $S(0)=0$ and

$$S(1) = 2(1-\ln 2)$$

3. (16points) Evaluate the volume integrals

$$\int dV r^2 \exp\left(i\overrightarrow{a}\cdot\overrightarrow{r}\right)\exp\left(-br^2\right)$$

and

$$\int dV r^2 \overrightarrow{r} \exp\left(i\overrightarrow{a}\cdot\overrightarrow{r}\right)\exp\left(-br^2\right)$$

$(dV = dxdydz)$. For the first integral, check your answer by considering the limiting case $a = 0$.

Solution

Consider the integral

$$
\begin{aligned}
I &= \int dV \exp\left(i \overrightarrow{a} \cdot \overrightarrow{r}\right) \exp\left(-br^2\right) = 2\pi \int_0^\infty dr r^2 \exp\left(-br^2\right) \int_{-1}^{1} d\left(\cos\theta\right) \exp\left(iar\cos\theta\right) \\
&= 2\pi \int_0^\infty dr r^2 \exp\left(-br^2\right) \frac{\exp\left(iar\right) - \exp\left(-iar\right)}{iar} = \frac{4\pi}{a} \int_0^\infty dr r \exp\left(-br^2\right) \sin\left(ar\right) \\
&= -\frac{4\pi}{a}\frac{\partial}{\partial a} \int_0^\infty dr \exp\left(-br^2\right) \cos\left(ar\right) = -\frac{4\pi}{a}\frac{\partial}{\partial a} \mathrm{Re}\left\{\int_0^\infty dr \exp\left(-br^2 + iar\right)\right\} \\
&= -\frac{4\pi}{a}\frac{\partial}{\partial a} \mathrm{Re}\left\{\int_0^\infty dr \exp\left[-b\left(r - i\frac{a}{2b}\right)^2 - \frac{a^2}{4b}\right]\right\} = -\frac{2\pi^{3/2}}{ab^{1/2}}\frac{\partial}{\partial a}\exp\left[-\frac{a^2}{4b}\right] = \left(\frac{\pi}{b}\right)^{3/2} \exp\left(-\frac{a^2}{4b}\right)
\end{aligned}
$$

Then

$$
\int dV r^2 \exp\left(i \overrightarrow{a} \cdot \overrightarrow{r}\right) \exp\left(-br^2\right) = -\frac{\partial I}{\partial b} = \left(\frac{\pi}{b}\right)^{3/2} \left[\frac{3}{2b} - \left(\frac{a}{2b}\right)^2\right] \exp\left(-\frac{a^2}{4b}\right)
$$

- Check: consider $a = 0$

$$
\begin{aligned}
\int dV r^2 \exp\left(-br^2\right) &= \int dV r^2 \exp\left(-br^2\right) = 4\pi \int_0^\infty dr r^4 \exp\left(-br^2\right) = 4\pi \frac{\partial^2}{\partial b^2} \int_0^\infty dr \exp\left(-br^2\right) \\
&= 4\pi \frac{\partial^2}{\partial b^2} \frac{\pi^{1/2}}{2b^{1/2}} = \frac{4}{3}\frac{1}{2}\frac{3}{2}\frac{\pi^{3/2}}{2b^{5/2}} = \left(\frac{\pi}{b}\right)^{3/2} \frac{3}{2b}
\end{aligned}
$$

Also

$$
\int dV r^2 \overrightarrow{r} \exp\left(i \overrightarrow{a} \cdot \overrightarrow{r}\right) \exp\left(-br^2\right) = -i\frac{\partial}{\partial \overrightarrow{a}} \left(-\frac{\partial I}{\partial b}\right) = i\left(\frac{\pi}{b}\right)^{3/2} \frac{\overrightarrow{a}}{2b} \left[\frac{5}{2b} - \left(\frac{a}{2b}\right)^2\right] \exp\left(-\frac{a^2}{4b}\right)
$$