MathPhys 15-Phys-721
Winter 2002 Final
Friday, March 15

1. (12 points) Solve the 2D Poisson equation inside a circle of radius a charged with the uniform charge density σ

$$\nabla_2^2 \phi = -4\pi\sigma$$

and find the electric field at the perimeter.

Hint: the radial part of the Laplace operator is

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)$$

Solution

Due to circular symmetry, the Poisson equation becomes

$$\begin{aligned} \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi}{d\rho}\right) &= -4\pi\sigma \\ \rho\frac{d\phi}{d\rho} &= -4\pi\sigma\frac{\rho^2}{2} + C_1 \\ \phi(\rho) &= -\pi\sigma\rho^2 + C_1 \ln\rho + C_2 = -\pi\sigma\rho^2 + C_2 \end{aligned}$$

since $\phi(0)$ must be finite. The electric field is given by

$$E = -\widehat{\boldsymbol{\rho}}\frac{d\phi(\rho)}{d\rho} = -2\pi\sigma a\widehat{\boldsymbol{\rho}}$$

2. (15 points) The steady-state concentration of a gas is described by the equation

$$\nabla^2 n - \varkappa^2 n = 0$$

Find n inside a sphere of radius a, given that

$$n(r = a) = n_0$$

Hint: the radial part of the Laplace operator is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$$

Solution

Due to spherical symmetry, the steady-state equation becomes

$$\begin{aligned} \frac{d^2 n}{dr^2} + \frac{2}{r}\frac{dn}{dr} - \varkappa^2 n &= 0 \\ r\frac{d^2 n}{dr^2} + 2\frac{dn}{dr} - \varkappa^2 nr &= 0 \\ \frac{d^2}{dr^2}(nr) - \varkappa^2 nr &= 0 \end{aligned}$$

whereof

$$nr = Ae^{\varkappa r} + Be^{-\varkappa r} = A\left(e^{\varkappa r} - e^{-\varkappa r}\right) = 2A\sinh(\varkappa r)$$

since $n(r)$ has to be finite at $r = 0$. From $n(r = a) = n_0$, find

$$n = n_0\frac{a}{r}\frac{\sinh(\varkappa r)}{\sinh(\varkappa a)}$$

3. (18 points) A conducting layer $0 \le x \le l$ was free from electromagnetic fields for $t < 0$. At time $t = 0$, a constant homogeneous magnetic field H_0 is developed everywhere outside the layer. Find the magnetic field in the layer for $t > 0$ and, in particular, at long times $t \gg l^2/\pi^2 a^2$. The equation for the magnetic field is given by

$$\frac{\partial^2 H}{\partial x^2} = \frac{4\pi\sigma\mu}{c^2}\frac{\partial H}{\partial t} \equiv a^{-2}\frac{\partial H}{\partial t}$$

where σ and μ are the electric conductivity and magnetic permeability respectively.

Solution

The boundary and initial conditions can be written as

$$\begin{aligned} H(0,t) &= H(l,t) = H_0 \\ H(x,0) &= 0,\ 0 \le x \le l \end{aligned}$$

Look for a solution as $h = H - H_0 \propto \exp(-\lambda t)$,

$$a^2 \frac{\partial^2 h}{\partial x^2} + \lambda h = 0$$

Subject to b.c.

$$h_n \propto \sin\frac{n x \pi}{l};\ \lambda_n = \left(\frac{n\pi a}{l}\right)^2$$

and

$$H(x,t) = H_0 + \sum_{n=1}^{\infty} A_n \exp\left[-\left(\frac{n\pi a}{l}\right)^2 t\right] \sin\frac{n x \pi}{l}$$

Subject to i.c.

$$0 = H_0 + \sum_{n=1}^{\infty} A_n \sin\frac{n x \pi}{l}$$

find

$$H(x,t) = H_0 + \sum_{k=0}^{\infty} \frac{\exp\left[-\frac{(2k+1)^2\pi^2 a^2}{l^2} t\right]}{2k+1} \sin\frac{(2k+1)x\pi}{l}$$

At long times, $t \gg l^2/\pi^2 a^2$,

$$H(x,t) \approx H_0 + \exp\left(-\frac{\pi^2 a^2}{l^2} t\right) \sin\frac{x\pi}{l}$$

4. (15 points) Find the transverse vibrations of a circular membrane

$$\frac{1}{r}\frac{d}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

fixed at the edge

$$u(r_0,t) = 0;\ 0 < t < \infty$$

and produced by a radially symmetric initial distributions of deflections and velocities

$$u(\mathbf{r},0) = \phi(r),\ \frac{\partial u}{\partial t}(\mathbf{r},0) = \psi(r);\ 0 < r < r_0$$

The zeros μ_n of $J_0(r)$ are assumed known. *Hint*:

$$\int_0^{r_0} J_0^2\left(\frac{\mu_n r}{r_0}\right) r dr = \frac{r_0^2}{2} J_1^2(\mu_n)$$

Solution

$$\begin{aligned} u(r,t) &= \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{\mu_n a t}{r_0}\right) + B_n \sin\left(\frac{\mu_n a t}{r_0}\right)\right] J_0\left(\frac{\mu_n r}{r_0}\right) \\ A_n &= \frac{2}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r\phi(r) J_0\left(\frac{\mu_n r}{r_0}\right) dr \\ B_n &= \frac{2}{a\mu_n r_0 J_1^2(\mu_n)} \int_0^{r_0} r\psi(r) J_0\left(\frac{\mu_n r}{r_0}\right) dr \end{aligned}$$