MathPhys 15-Phys-721
Fall 2001 Midterm

1. (14 points) Using contour integration evaluate the integral

$$I=\int_0^\infty dx\frac{\sqrt{x}}{1+x^4}$$

Hint: think carefully about the contour you want to use.

Solution

Consider the integral

$$J=\oint dz\frac{\sqrt{z}}{1+z^4}$$

along the a closed contour consisting of *(i)* the real axis from 0 to ∞, *(ii)* One quarter of a large circle at $|z|=\infty$, and *(iii)* a return to the origin along the line $\arg z=\pi/2$ (imaginary axis). Then

$$\begin{aligned}I\left(1-e^{3i\pi/4}\right) &= J=2\pi i\frac{e^{i\pi/8}}{\left(e^{i\pi/4}-e^{3i\pi/4}\right)\left(e^{i\pi/4}-e^{5i\pi/4}\right)\left(e^{i\pi/4}-e^{7i\pi/4}\right)}\\ &= 2\pi\frac{e^{5i\pi/8}}{e^{3i\pi/4}\left(1-e^{i\pi/2}\right)\left(1-e^{i\pi}\right)\left(1-e^{3i\pi/2}\right)}=\frac{\pi}{2e^{i\pi/8}}\end{aligned}$$

On the other hand

$$I\left(1-e^{3i\pi/4}\right)=Ie^{3i\pi/8}\left(e^{-3i\pi/8}-e^{3i\pi/8}\right)=-2iIe^{3i\pi/8}\sin\frac{3\pi}{8}$$

so that

$$I=\frac{\pi}{4\sin\frac{3\pi}{8}}$$

2. (13 points) Consider a 2π-periodic function

$$f(x)=\frac{x}{2}-\frac{x^2}{4\pi},\ x\in[0,2\pi)$$

- Expand $f(x)$ into Fourier series

$$f(x)=\sum_{k=-\infty}^{\infty}a_k\exp(ikx)$$

  and determine a_k.
- Differentiate both sides twice to obtain the Poisson summation formula.

Solution

$$f(x)=\frac{\pi}{6}-\frac{1}{2\pi}\sum_{k\neq 0}\frac{\exp(ikx)}{k^2}$$

Differentiating twice,

$$-\frac{1}{2\pi}+\sum_{n=-\infty}^{\infty}\delta(x-2n\pi)=\frac{1}{2\pi}\sum_{k\neq 0}\exp(ikx)$$

one obtains the Poisson summation formula

$$\sum_{n=-\infty}^{\infty}\delta(x-2n\pi)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\exp(ikx)$$

3. (12 points) In atomic units, the normalized radial function of the ground state of a hydrogen atom is

$$\psi(r) = 2\exp(-r)$$

Find its Fourier transform.

Solution

$$\begin{aligned}\psi(k) &= \int d^3x\psi(r)\exp\left(-i\overrightarrow{k}\cdot\overrightarrow{r}\right) = 2\pi\int_0^\infty drr^2\psi(r)\int_{-1}^1 d\alpha\exp(-ikr\alpha) \\ &= \frac{4\pi}{k}\int_0^\infty drr\psi(r)\sin(kr) = \frac{8\pi}{k}\,\mathrm{Im}\int_0^\infty drr\exp[-r(1-ik)] \\ &= \frac{8\pi}{k}\,\mathrm{Im}\frac{1}{(1-ik)^2} = \frac{16\pi}{(1+k^2)^2}\end{aligned}$$

4. (10 points) For a collection of small metal particles the variations in the position of the Fermi level from particle to particle results in a correction to the Pauli spin susceptibility relative to its value in the bulk metals. The evaluation of such a correction reduces to the evaluation of the integral

$$I = \int_{-\infty}^\infty \frac{d\omega}{(\omega+i\gamma)^4}\frac{\omega}{\exp(\beta\omega)-1}$$

in the limit $\gamma \to 0$ (the role of γ is solely to underscore the fact that there is no pole at $\omega = 0$). Find I using $\sum_{m=1}^\infty m^{-3} = \zeta(3)$.

Solution

The poles are at $\omega_m = 2\pi im/\beta$, $m > 0$ with the contour of integration closed in the upper plane. Consequently,

$$I = 2\pi i\sum_{m=1}^\infty \frac{1}{(2\pi im/\beta)^3}\frac{1}{\beta} = -\left(\frac{\beta}{2\pi}\right)^2\zeta(3)$$

5. (11 points) The Fresnel integral $S(x)$ is defined as

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$$

Find the first two terms in the expansion of $S(x)$ for small x and for large x.

Solution

For small x,

$$S(x) \approx \int_0^x \frac{\pi t^2}{2}dt = \frac{\pi x^3}{6}$$

For large x,

$$\begin{aligned}S(x) &= S(\infty) - \int_x^\infty \sin\left(\frac{\pi t^2}{2}\right)dt = S(\infty) - \sqrt{\frac{1}{2\pi}}\int_{\pi x^2/2}^\infty \frac{\sin(y)}{\sqrt{y}}dy \\ &= S(\infty) - \sqrt{\frac{1}{2\pi}}\frac{\cos(\pi x^2/2)}{\sqrt{\pi x^2/2}} + \frac{1}{2}\sqrt{\frac{1}{2\pi}}\int_{\pi x^2/2}^\infty \frac{\cos(y)}{y^{3/2}}dy \\ &= S(\infty) - \frac{\cos(\pi x^2/2)}{\pi x} + O\left[\frac{1}{x}\right]^2 \approx \frac{1}{2} - \frac{\cos(\pi x^2/2)}{\pi x}\end{aligned}$$

since

$$S(\infty) = \int_0^\infty \sin\left(\frac{\pi t^2}{2}\right)dt = -\,\mathrm{Im}\int_0^\infty \exp\left(-i\frac{\pi t^2}{2}\right)dt = -\,\mathrm{Im}\frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{i\frac{\pi}{2}}} = \frac{1}{2}$$