13.4 (see Problem 2, LL §11, p. 27)

(a) $U = -\dfrac{U_0}{\cosh^2 \alpha x}$, $T = \dfrac{\pi}{\alpha}\sqrt{\dfrac{2m}{-E}}$

$$T = 2\pi \frac{\partial I}{\partial E} = \frac{\pi}{\alpha}\sqrt{\frac{2m}{-E}}$$

$$\frac{\partial I}{\partial E} = \frac{1}{2\alpha}\sqrt{\frac{2m}{-E}}, \quad I = -\frac{1}{\alpha}\sqrt{-2mE} + C$$

$$E = -U_0, \quad I = 0, \quad C = \frac{1}{\alpha}\sqrt{2mU_0}$$

$$I = \frac{\sqrt{2m}}{\alpha}\left(\sqrt{U_0} - \sqrt{|E|}\right)$$

(b) $U = U_0 \tan^2 \alpha x$, $T = \dfrac{\pi}{\alpha}\sqrt{\dfrac{2m}{U_0 + E}}$

$$\frac{\partial I}{\partial E} = \frac{1}{2\alpha}\sqrt{\frac{2m}{E+U_0}}, \quad I = \frac{\sqrt{2m}}{\alpha}\left(\sqrt{E+U_0} + \text{const}\right)$$

$$I = \frac{\sqrt{2m}}{\alpha}\left(\sqrt{E+U_0} - \sqrt{U_0}\right), \text{ since } I(E=0) = 0$$

(c) $U = A|x|^n$, $T = \dfrac{2}{n}\sqrt{\dfrac{2\pi m}{E}}\left(\dfrac{E}{A}\right)^{1/n}\dfrac{\Gamma(1/n)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$

$$I = \frac{1}{\pi n}\sqrt{2\pi m E}\left(\frac{E}{A}\right)^{\frac{1}{n}}\frac{\Gamma\left(\frac{1}{n}\right)}{\left(\frac{1}{2}+\frac{1}{n}\right)\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

$$= \frac{1}{n}\sqrt{\frac{2mE}{\pi}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{n}\right)}$$

constants = {$\omega_0 \to 2\pi$, $\varepsilon \to 1/(4\pi)$, $P_0 \to 1$};

ω[t_] := ω_0 + Sin[$(\varepsilon\,\omega_0)^2$ t] /. constants;

$\dot{\omega}_0 = \partial_t\ \omega$[t] /. t → 0;

$t_{upp} = \dfrac{4}{\varepsilon\ \omega_0}$ /.constants;

Q_0[t_] := ω_0 t /.constants

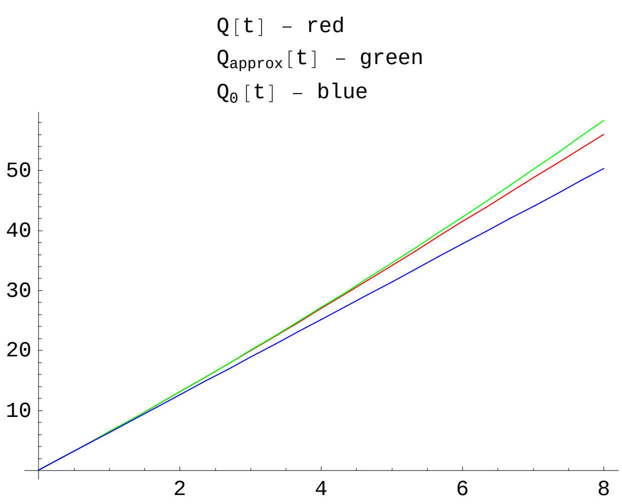
Q_{approx}[t_] := Q_0[t] + $\dfrac{1}{2}\dot{\omega}_0\, t^2 + \dfrac{\dot{\omega}_0}{2\ \omega_0}\displaystyle\int_0^t$ Sin[2 $\omega_0\ \tau$] $d\tau$ /.constants

P_{approx}[t_] := $P_0 \left(1 - \dfrac{\dot{\omega}_0}{\omega_0}\displaystyle\int_0^t \text{Cos}[2\,\omega_0\,\tau\,]\,d\tau\right)$ /.constants

**nds = NDSolve[{∂_t Q[t] == ω[t] + $\dfrac{\partial_t\ \omega[t]}{2\,\omega[t]}$ Sin[2Q[t]], ∂_t P[t] == −P[t] $\dfrac{\partial_t\ \omega[t]}{\omega[t]}$ Cos[2Q[t]], Q[0] == 0, P[0] == P_0 /.constants}, {Q, P},
{t, 0, t_{upp}}, MaxSteps → 3000]**

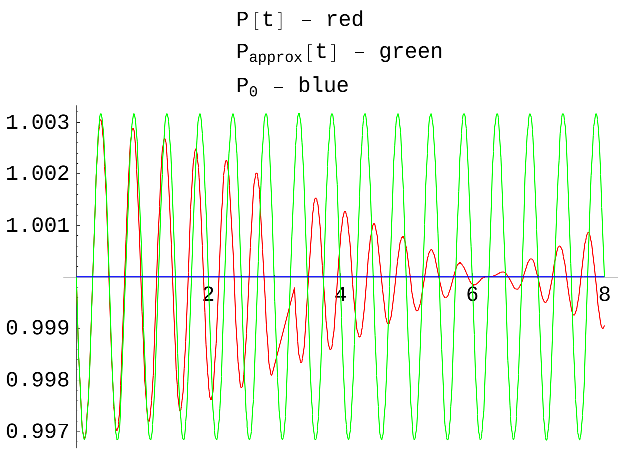
```
{{Q → InterpolatingFunction[{{0., 8.}}, <>], P → InterpolatingFunction[{{0., 8.}}, <>]}}
```

**plotQ =
Plot[
{Evaluate [Q[t] /.nds], Q_{approx}[t], Q_0[t]}, {t, 0, t_{upp}}, PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]}, PlotLabel → "\<Q[t] − red
Q_{approx}[t] − green
Q_0[t] − blue\>", ImageSize → 288]**



- Graphics -

**plotP = Plot[{Evaluate [P[t] /. nds], P_{approx}[t], P_0 /. constants},
{t, 0, t_{upp}}, PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0], RGBColor[0, 0, 1]}, PlotLabel → "\<P[t] – red
P_{approx}[t] – green
P_0 – blue\>", ImageSize → 288]**

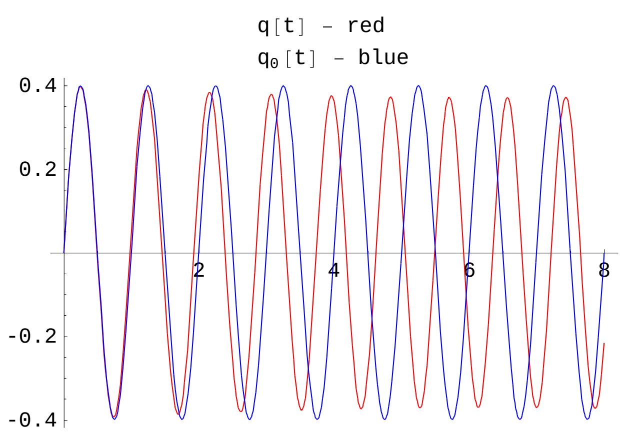


- Graphics -

q[t_] := $\sqrt{\dfrac{\text{Evaluate[P[t] /.nds]}}{\omega[\text{t}]}}$ Sin[Evaluate[Q[t] /. nds]]

q_0[t_] := $\sqrt{\dfrac{P_0}{\omega_0}}$ Sin[ω_0 t] /.constants

**Plot[{q[t], q_0[t]}, {t, 0, t_{upp}}, PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}, PlotLabel → "\<q[t] – red
q_0[t] – blue\>", ImageSize → 288]**



- Graphics -

13.10

$$\ddot{q} = \ddot{\left(\frac{1}{\omega^{1/2}}\right)} \exp(\,) + 2 \dot{\left(\frac{1}{\omega^{1/2}}\right)} \dot{\left[\exp(\,)\right]} + \frac{1}{\omega^{1/2}} \ddot{\left[\exp(\,)\right]}$$

$$= \dot{\left(-\frac{\dot{\omega}}{2\omega^{3/2}}\right)} \exp + 2\left(-\frac{\dot{\omega}}{2\omega^{3/2}}\right)(i\omega)\exp + \frac{1}{\omega^{1/2}}\left[(i\omega)\dot{\left(\exp(\,)\right)}\right]$$

$$= \left(-\frac{\ddot{\omega}}{2\omega^{3/2}} + \frac{3\dot{\omega}^2}{4\omega^{5/2}}\right)\exp - \cancel{\frac{i\dot{\omega}}{2\omega^{1/2}}\exp} + \cancel{\frac{i\dot{\omega}}{\omega^{1/2}}\exp} - \frac{\omega^2}{\omega^{1/2}}\exp$$

$$\ddot{q} + \omega^2 q = \left(-\frac{\ddot{\omega}}{2\omega^{3/2}} + \frac{3\dot{\omega}^2}{4\omega^{5/2}}\right)\exp$$

$$\ll \frac{\dot{\omega}}{\omega^{1/2}}\exp \ll \omega^{3/2}\exp \sim \omega^2 q$$

13.11

$$x = \frac{F}{m\omega^2} + \sqrt{\frac{2I}{m\omega}} \sin w \equiv X + A \sin w \qquad (1)$$

$$\dot{I} = -\left(\frac{\partial \Lambda}{\partial w}\right)_{I,F} \dot{F} \text{, where}$$

$$\Lambda = \left(\frac{\partial S_0}{\partial F}\right)_{x,I} = \left(\frac{\partial S}{\partial w}\right)_I \left(\frac{\partial w}{\partial F}\right)_x$$

$$\frac{\partial x}{\partial F} = 0 \overset{\text{using (1)}}{=} \frac{\partial X}{\partial F} + A \cos w \left(\frac{\partial w}{\partial F}\right)_x \text{ : solve for } \left(\frac{\partial w}{\partial F}\right)_x$$

see LL p. 159

$$\Lambda = 2I \cos^2 w \left(\frac{\partial w}{\partial F}\right)_x = -\frac{2I \cos w}{A m \omega^2}$$

use relationship between A and I

$$\dot{I} = -\frac{A}{\omega} \sin w \dot{F} = -\frac{A}{\omega} \sin(\omega t + \varphi) \dot{F}$$

$$w = \omega t + \varphi + \ldots$$

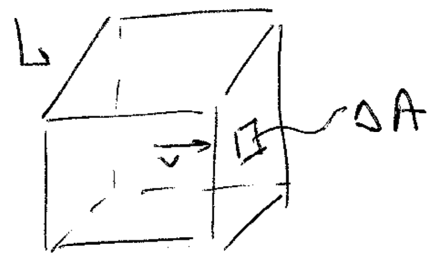
$$x(0) = \frac{F(0)}{m\omega^2} + A \sin\varphi \;, \quad \dot{x}(0) = A\omega \cos\varphi$$

$$\dot{I} = -\frac{\dot{x}(0)}{\omega^2} \sin \omega t \dot{F}(t) - \frac{1}{\omega}\left[x(0) - \frac{F(0)}{m\omega^2}\right] \dot{F}(t)$$

$$I = I(0) - \frac{\dot{x}(0)}{\omega^2} \int_0^t \sin \omega t \dot{F}(t) d\tau$$

$$- \frac{1}{\omega}\left[x(0) - \frac{F(0)}{m\omega^2}\right] \int \cos \omega t \dot{F}(t) dt$$

13, 12



$$p\,\Delta A = F$$
$$F\,\Delta t = \mathcal{N}\, 2mv$$
$$\mathcal{N} = n\,v\,\Delta t\,\Delta A$$

(1) $p = 2mv^2 n = 2mN\,\frac{v^2}{V}$

(2) $I = \oint p\,dq = 2mvL = 2mvV^{1/3}$

From (1), $v \propto (pV)^{1/2}$

From (2), $vV^{1/3} \approx \text{const}$

Combining the two $p^{1/2}V^{5/6} \approx \text{const}$

adiabadic process $pV^{5/3} \approx \text{const}$

13.13(a)

$$\left.\begin{aligned} I_x &= \frac{1}{2\pi} m v_x 2a = \frac{m}{\pi} v_x a \\ I_y &= \frac{m}{\pi} v_y b \\ I_z &= \frac{m}{\pi} v_z c \end{aligned}\right\} \text{adiabatic invariant}$$

$$\left.\begin{aligned} \omega_x &= \frac{\pi v_x}{a} \\ \omega_y &= \frac{\pi v_y}{b} \\ \omega_z &= \frac{\pi v_z}{c} \end{aligned}\right\} \text{quasiperiodic motion}$$

$$E = \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) = \frac{\pi^2}{2m}\left(\frac{I_x^2}{a^2} + \frac{I_y^2}{b^2} + \frac{I_z^2}{c^2}\right)$$

Semiclassical quantization

$$I_x \simeq \hbar l, \quad I_y \simeq \hbar m, \quad I_z \simeq \hbar n$$

$$E_{lmn} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

13.15 (a)

$$I = \frac{1}{2\pi}\oint dr\sqrt{\underbrace{E - \frac{M^2}{2mr^2} - U(r)}_{-U_{eff}(r)}} \quad \left(= \frac{1}{\pi}\int_{r_{min}}^{r_{max}}\right)$$

$= \text{const}$

$U \to U + \delta U$, $E \to E + \delta E$, see 13.17

$$\delta E = \langle \delta U \rangle = \frac{\oint \frac{\delta U(r)\, dr}{v(r)}}{\int \frac{dr}{v(r)}} = \frac{1}{T}\oint \frac{\delta U(r)\, dr}{v(r)}$$

Here $\delta U = \frac{\delta\gamma}{r^n} = \frac{\delta\gamma}{\gamma}\frac{\gamma}{r^n} = \frac{\delta\gamma}{\gamma} U$

$$\delta E = \frac{\delta\gamma}{\gamma}\langle U \rangle = \frac{\delta\gamma}{\gamma}\frac{2E}{-n+2},$$

by virial theorem, see LL §10

$$\frac{\partial E}{\partial \gamma} = \frac{E}{\gamma}\frac{2}{-n+2}, \qquad E \propto \gamma^{2/2-n}$$

Check: consider almost circular motion, $r \approx r_{bott}$

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{r=r_{bott}} = 0, \quad r_{bott} \propto \gamma^{1/n-2}$$

$$E \sim U_{eff}(r_{bott}) \propto \frac{1}{r_{bott}^2} \propto \gamma^{2/2-n}$$