Kepler Problem

$$S = -Et + p_\phi \phi + \int dr \sqrt{2m\left(E - \frac{\alpha}{r}\right) - \frac{p_\phi^2}{r^2}}$$

a) $\frac{\partial S}{\partial p_\phi} = \text{const} \equiv \phi_0 = 0$, without loss of generality

$$0 = \phi - \int \frac{p_\phi \frac{dr}{r^2}}{\sqrt{2m\left(E - \frac{\alpha}{r}\right) - \frac{p_\phi^2}{r^2}}}$$

$$\phi = \cos^{-1} \frac{\frac{p_\phi}{r} - \frac{m\alpha}{p_\phi}}{\sqrt{2mE + \frac{m^2\alpha^2}{p_\phi^2}}} \quad \text{or} \quad \frac{p}{r} = 1 + \varepsilon \cos\phi$$

$$p = \frac{p_\phi^2}{m\alpha}, \quad \varepsilon = \sqrt{1 + \frac{2E p_\phi^2}{m\alpha^2}}$$

b) $\frac{\partial S}{\partial E} = \text{const} = -t + \int \frac{m}{\sqrt{2m\left(E - \frac{\alpha}{r}\right) - \frac{p_\phi^2}{r^2}}} dr$

Kepler Problem

Consider $E = -\frac{m\alpha^2}{2p_\phi^2}$, $\varepsilon = 0$, $r = p$

circular motion

$$S = \frac{m\alpha^2}{2p_\phi^2} t + p_\phi \phi$$

$$\frac{\partial S}{\partial p_\phi} = \text{const} = -\frac{m\alpha^2}{p_\phi^3} t + \phi$$

const $= \phi_0 = 0$, without loss of generality

$$\phi = \frac{m\alpha^2}{p_\phi^3} t \qquad \underline{r} = p = \frac{p_\phi^2}{m\alpha}, \quad \alpha^2 = \frac{p_\phi^4}{m^2 p^2}$$

$$= \frac{p_\phi}{mr^2} t$$

Alternatively

$$S = -Et + \sqrt{-\frac{m\alpha^2}{2E}}\,\phi$$

$$\frac{\partial S}{\partial E} = \text{const} = 0 = -t + \sqrt{\frac{m\alpha^2}{2}} \frac{1}{2(-E)^{3/2}} \phi$$

$$= -t + \frac{p_\phi^3}{m\alpha^2}\phi, \text{ as before}$$