Problem 4.9

$$\mathcal{L} = \frac{1}{2} m_a v_a^2 - U(\vec{r}_a, t)$$

$$\vec{v} = \vec{v}_a' + \vec{V}, \quad \vec{r}_a = \vec{r}_a' + \vec{V} t$$

$$\mathcal{L}_1' = \frac{1}{2} m_a v_a'^2 + m_a \vec{v}_a' \cdot \vec{V} + \frac{1}{2} \mu \vec{V}^2 - U(\vec{r}_a' + \vec{V} t, t)$$

$$= \frac{1}{2} m_a v_a'^2 - U(\vec{r}_a' + \vec{V} t, t)$$

$$+ \vec{P}' \cdot \vec{V} + \frac{1}{2} \mu \vec{V}^2 = \mathcal{L}_2' + \vec{P}' \cdot \vec{V} + \frac{1}{2} \mu \vec{V}^2$$

$$\mathcal{L}_2' + \left(\mu \vec{\dot{R}}' \cdot \vec{V} + \frac{1}{2} \mu \vec{V}^2 t \right)$$

For purposes of 4.10, 4.11 and 4.14

$\vec{r}, \vec{t} \to \vec{r}', t'$ implies $\mathcal{L} \to \mathcal{L}' + \frac{df'}{dt'}$

$$f' = \mu \vec{R}' \cdot \vec{V} + \frac{1}{2} \mu \vec{V}^2 t$$

(a) $$E_1' = \frac{\partial \mathcal{L}_1'}{\partial \dot{\vec{r}}_a'} \dot{\vec{r}}_a' - \mathcal{L}_1' \Big|_{\vec{r}_a' = \vec{r} - \vec{V} t}$$

$$= \left(m_a \vec{v}_a' + m_a \vec{V} \right) \cdot \vec{v}_a' - \mathcal{L}_1'$$

$$= \frac{1}{2} m_a \vec{v}_a'^2 + U(\vec{r}_a' + \vec{V} t, t)$$

$$- \frac{1}{2} \mu \vec{V}^2$$

(continued)

Problem 4.9 (continued)

$$= \frac{1}{2} m_a (\vec{v}_a - \vec{V})^2 - \frac{1}{2} \mu V^2 + U(\vec{r}, t)$$

$$= \frac{1}{2} m_a v_a^2 - \vec{P} \cdot \vec{V} + U(\vec{r}, t) + \cancel{\frac{1}{2} \mu V^2} - \cancel{\frac{1}{2} \mu V^2}$$

$$\vec{P}_1' = m_a \vec{v}_a' + \mu \vec{V} = m_a (\vec{v}_a - \vec{V}) + \mu \vec{V}$$
$$= \vec{P}$$

(b) $$\vec{P}_2' = \frac{\partial \mathcal{L}_2'}{\partial \vec{v}_a'} = m_a \vec{v}_a' = m_a \vec{v}_a - \mu \vec{V}$$
$$= \vec{P} - \mu \vec{V}$$

$$E_2' = \frac{\partial \mathcal{L}_2'}{\partial \vec{v}_a'} \cdot \vec{v}_a' - \mathcal{L}_2'$$

$$= \frac{1}{2} m_a \vec{v}_a'^2 + U(\vec{r}' + \vec{V}t, t)$$

$$= \frac{1}{2} m_a (\vec{v}_a - \vec{V})^2 + U(\vec{r}, t)$$

$$= E + \frac{1}{2} \mu V^2 - \vec{P} \cdot \vec{V}$$

Compare LL (8.5)

$$E = E' + \vec{V} \cdot \vec{P}' + \frac{1}{2} \mu V^2$$

$$E' = E - \vec{V} \cdot \vec{P}' - \frac{1}{2} \mu V^2$$

$$= E - \vec{V} \cdot (\vec{P} - \mu \vec{V}) - \frac{1}{2} \mu V^2 = E + \frac{1}{2} \mu V^2 - \vec{V} \cdot \vec{P}$$

Problem 4.10

$$\mathcal{L}'dt' - \mathcal{L}dt = (\mathcal{L}' - \mathcal{L})dt' + \mathcal{L}(dt' - dt)$$

$$= \delta\mathcal{L}\,dt' + \mathcal{L}\,\delta dt \simeq \delta\mathcal{L}\,dt + \mathcal{L}\,\delta dt$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i + \frac{\partial\mathcal{L}}{\partial t}\delta t = \dot{p}_i\delta q_i + p_i\delta\dot{q}_i - \dot{E}\delta t$$

$$\delta q_i = \varepsilon\psi_i \;;\quad \delta t = \varepsilon X \;;\quad \delta dt = \varepsilon dX$$

$$\dot{q}_i' = \frac{dq_i'}{dt'} = \frac{dq_i + \varepsilon d\psi_i}{dt + \varepsilon dX} = \frac{dq_i}{dt}\left(1 + \varepsilon\frac{d\psi_i}{dq_i}\right)\left(1 - \varepsilon\frac{dX}{dt}\right)$$

$$\delta\dot{q}_i = \varepsilon\frac{d\psi_i}{dt} - \varepsilon\frac{dq_i}{dt}\frac{dx}{dt}$$

$$\delta\mathcal{L} = \dot{p}_i\,\varepsilon\psi_i + p_i\,\varepsilon\frac{d\psi_i}{dt} - p_i\,\varepsilon\frac{dq_i}{dt}\frac{dx}{dt}$$

$$\delta S = dt\left[(p_i\psi_i)^{\cdot} - p_i\dot{q}_i\dot{X} - \dot{E}X + \mathcal{L}\dot{X}\right]\varepsilon = 0$$

$$(p_i\psi_i)^{\cdot} - \underbrace{(p_i\dot{q}_i - \mathcal{L})}_{E}\dot{X} - \dot{E}X = 0$$

$$(p_i\psi_i)^{\cdot} - (EX)^{\cdot} = 0$$

$$p_i\psi_i - EX = \text{const}$$

Problem 4.11: $\quad p_i\psi_i - EX + f = \text{const}$

Problem 4.12 (c)

$$\left.\begin{array}{l} x = x'\cosh\lambda + t'\sinh\lambda \\ t = x'\sinh\lambda + t'\cosh\lambda \end{array}\right\} \lambda : \text{small} \Rightarrow \left\{\begin{array}{l} x \simeq x' + t'\lambda \\ t = t' + x'\lambda \end{array}\right.$$

$$\left.\begin{array}{l} x' = x\cosh\lambda - t\sinh\lambda \\ t' = t\cosh\lambda - x\sinh\lambda \end{array}\right\} \lambda : \text{small} \Rightarrow \left\{\begin{array}{l} x' \simeq x - t\lambda \\ t' = t - x\lambda \end{array}\right.$$

$$\delta x = x' - x = -t\lambda \qquad dt' = dt - dx\,\lambda$$

$$\delta t = t' - t = -x\lambda \qquad \delta dt = -dx\,\lambda$$

$$\left.\begin{array}{l} dx' = dx - \lambda dt \\ dt' = dt - \lambda dx \end{array}\right\} \frac{dx'}{dt'} = \frac{dx}{dt}\left(1 - \lambda\frac{dt}{dx}\right)\left(1 + \lambda\frac{dx}{dt}\right)$$

$$\delta\dot{x} = -\lambda + \lambda\dot{x}^2$$

$$\overbrace{dS' - dS}^{\delta dS} = \mathcal{L}'dt' - \mathcal{L}dt \simeq \delta\mathcal{L}\,dt + \mathcal{L}\,\delta dt = 0$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial\dot{x}}\delta\dot{x} + \frac{\partial\mathcal{L}}{\partial t}\delta t = \dot{p}_x\delta x + p_x\delta\dot{x} - \dot{E}\delta t$$

$$= -\dot{p}_x t\lambda - p_x\lambda + p_x\dot{x}^2\lambda + \dot{E}x\lambda$$

$$\left(-\dot{p}_x t - p_x + p_x\dot{x}^2 + \dot{E}x\right)\lambda\,dt + \mathcal{L}\left(-dx\,\lambda\right) = 0$$

$$-\dot{p}_x t - p_x + p_x\dot{x}^2 + \dot{E}x - \mathcal{L}\dot{x} = 0$$

$$(-p_x t)^{\cdot} + \dot{E}x + \dot{x}\left(p_x\dot{x} - \mathcal{L}\right) = 0$$

$$(-p_x t)^{\cdot} + \dot{E}x + \dot{x}E = 0 \qquad \left(-p_x t + Ex\right)^{\cdot} = 0$$

$$p_x t - Ex = \text{const}$$

Problem 4.13 (b)

$$S \propto \mathcal{L} t \propto \alpha^n \beta \sim \alpha^n \alpha^{1-\frac{1}{2}n} \sim \alpha^{1+\frac{1}{2}n}$$

$1+\frac{1}{2}n = 0$: action unchanged under
$n=-2$ similarity transformation

$$\vec{r}\,' = \vec{r}\alpha = (1+\varepsilon)\vec{r}, \quad \delta\vec{r} = \varepsilon\vec{r}$$

$$t' = t\beta = t\alpha^{(1-\frac{1}{2}n)} \simeq t\left(1+\frac{2-n}{2}\varepsilon\right)$$

$$\delta t = \frac{2-n}{2}\varepsilon t$$

$$d\delta t = \delta dt = \frac{2-n}{2}\varepsilon\, dt$$

$$\frac{d\vec{r}\,'}{dt'} = \frac{d\vec{r}(1+\varepsilon)}{dt\left(1+\frac{2-n}{2}\varepsilon\right)} \simeq \frac{d\vec{r}}{dt}\left(1+\varepsilon+\frac{n-2}{2}\varepsilon\right) = \frac{d\vec{r}}{dt}\left(1+\frac{n}{2}\varepsilon\right)$$

$$\delta\dot{\vec{r}} = \frac{n}{2}\varepsilon\dot{\vec{r}}$$

$$\delta dS = 0 = \delta\mathcal{L}\, dt + \mathcal{L}\,\delta dt$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\vec{r}}\delta\vec{r} + \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}}\delta\dot{\vec{r}} = \dot{\vec{p}}\,\delta\vec{r} + \vec{p}\,\delta\dot{\vec{r}}$$

$$= \dot{\vec{p}}\,\varepsilon\vec{r} + \vec{p}\,\frac{n}{2}\varepsilon\dot{\vec{r}}$$

$$0 = \left(\dot{\vec{p}}\vec{r} + \vec{p}\,\frac{n}{2}\dot{\vec{r}} + \mathcal{L}\,\frac{2-n}{2}\right)\varepsilon\, dt$$

$$0 = \dot{\vec{p}}\vec{r} + \vec{p}\,\frac{n}{2}\dot{\vec{r}} + \mathcal{L}\,\frac{2-n}{2} \overset{n=-2}{=} \dot{\vec{p}}\cdot\vec{r} - \vec{p}\cdot\dot{\vec{r}} + 2\mathcal{L}$$

$$= \dot{\vec{p}}\cdot\vec{r} + \vec{p}\cdot\dot{\vec{r}} - 2(\vec{p}\cdot\dot{\vec{r}} - \mathcal{L}) = (\vec{p}\cdot\vec{r})^{\cdot} - 2(Et)^{\cdot} \;{}^{*}$$

* E = const $\qquad \Rightarrow \vec{p}\cdot\vec{r} - 2Et = \text{const}$

Problem 4.14

$\mathcal{L} \to \mathcal{L}' + \mu(\dot{\vec{R}}\cdot\vec{\varepsilon})$, taking $\vec{V} = \vec{\varepsilon}$

and keeping <u>leading</u> terms in $\vec{\varepsilon}$

Here $\mathcal{L}'$ corresponds to $\mathcal{L}_2'$ in (4.9)

Using notations of (4.11),

$\vec{\psi} = -t\vec{\varepsilon}/\varepsilon$, $f = \mu(\vec{R}\cdot\vec{\varepsilon})/\varepsilon$, and $\chi = 0$

and, according to (4.11)

$-\vec{P}\,t + \mu\vec{R} = \text{const}$,

$\dot{\vec{R}} = \dfrac{\dot{\vec{P}}}{\mu}t + \dfrac{\vec{P}}{\mu}$; accelerated motion of c.m.

Notice, that 4.12 (e) can be generalized to

$\vec{P}t - E\vec{R} = \text{const}$ (see LL2 §14, p. 42)

In non-relativistic limit, this yields

$\vec{P}t - \mu\vec{R} = \text{const}$, as above