.1

onsider $U(\vec{r}) = mgy$, as in gravity field

$$\frac{}{m}\left[\left(\frac{\partial S_0}{\partial x}\right)^2 + \left(\frac{\partial S_0}{\partial y}\right)^2\right] + mgy = E$$

$S_0 = p_{x0}x + S_{0y}$, since x is ignorable

$$\frac{1}{2m}\left[p_{x0}^2 + \left(\frac{\partial S_{0y}}{\partial y}\right)^2\right] + mgy = E$$

$$\frac{\partial S_{0y}}{\partial y} = \sqrt{2m(E-mgy) - p_{x0}^2}$$

ntegrating on y,

$$S = -\frac{\left[2m(E-mgy) - p_{x0}^2\right]^{3/2}}{3gm^2} + p_{x0}x - Et$$

$$\frac{\partial S}{\partial E} = c$$

$$-t - \frac{\left[2m(E-mgy) - p_{x0}^2\right]^{1/2}}{mg} = c$$

$$y = -g\frac{(t+c)^2}{2} - \left(\frac{p_{x0}}{m}\right)^2\frac{1}{2g} + \frac{E}{mg} \qquad (1)$$

itial conditions: $y(t=0) = 0$

2.1

$$0 = \frac{gc^2}{2} - \frac{v_{x0}^2}{2g} + \frac{E}{mg} = \frac{gc^2}{2} + \frac{v_{y0}^2}{2g}$$

$$E = \frac{m(v_{x0}^2 + v_{y0}^2)}{2}$$

(*)

$c = \pm\frac{v_{y0}}{g}$ and from Eqs. (1) and (*)

$$y = -g\frac{\left(t - \frac{v_{y0}}{g}\right)^2}{2} - \frac{v_{x0}^2}{2g} + \frac{v_{x0}^2 + v_{y0}^2}{2g}$$

$$= -\frac{1}{2}gt^2 + v_{y0}t$$

) $\frac{\partial S}{\partial p_{x0}} = b$

$$x + \frac{\sqrt{2m(E - mgy) - p_{x0}^2}\, p_{x0}}{gm^2} = b$$

$$x + \frac{\sqrt{v_{y0}^2 - 2gy}\, v_{x0}}{g} = b$$

$x = y = 0$: initial cond. $\to b = \frac{v_{y0} v_{x0}}{g}$

$$y = x\frac{v_{y0}}{v_{x0}} - \frac{x^2 g}{2v_{x0}^2}$$

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$$\mathbf{f}[\theta\_,\ \xi\_] = \int \frac{1}{\sqrt{1 - \xi\,\mathrm{Cos}[\theta]}}\, d\theta;$$

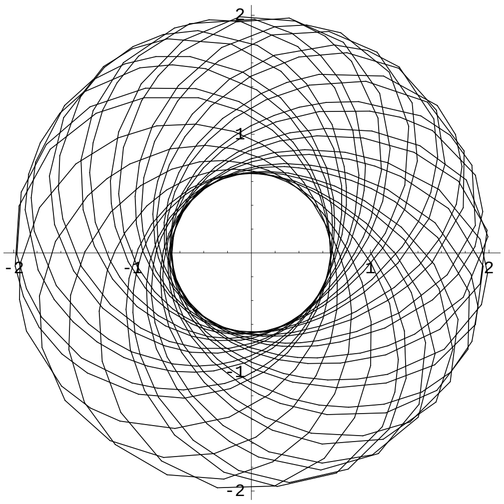
$$\mathbf{r}[\mathrm{p}\_,\ \varepsilon\_,\ \theta\_,\ \xi\_] = \frac{\mathrm{p}}{1 + \varepsilon\,\mathrm{Cos}[\mathrm{f}[\theta,\ \xi]]}$$

$$\frac{\mathrm{p}}{1 + \varepsilon\,\mathrm{Cos}\left[\frac{2\sqrt{\frac{1-\xi\,\mathrm{Cos}[\theta]}{1-\xi}}\ \mathrm{EllipticF}\left[\frac{\theta}{2}, -\frac{2\xi}{1-\xi}\right]}{\sqrt{1-\xi\,\mathrm{Cos}[\theta]}}\right]}$$

constants1 = {p → 1, ε → .5, ξ → .5};

r1 = r[p, ε, θ, ξ] /.constants1;

PolarPlot[r1, {θ, 0,64 π}]



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$$\mathbf{g} = \int_0^{2\pi} \frac{1}{\sqrt{1 - \xi\,\mathrm{Cos}[\theta]}}\, d\theta$$

$$\frac{4\,\mathrm{EllipticK}\left[-\frac{2\xi}{1-\xi}\right]}{\sqrt{1-\xi}}$$

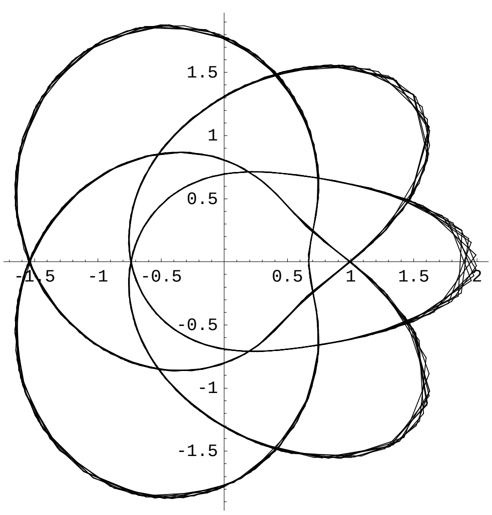
FindRoot[g == 5 π / 2, {ξ, {.6,.7}}]

{ξ → 0.847551}

constants2 = {p → 1, ε → .5, ξ → 0.847550910147551306};

r2 = r[p, ε, θ, ξ] /.constants2;

PolarPlot[r2, {θ, 0,64 π}]



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2.8 (b)

$$S = -Et + \int d\theta \sqrt{\beta - 2ma\cos\theta}$$

$$+ \int dr \sqrt{2m\left[E - \frac{\gamma}{r^4}\right] - \frac{\beta}{r^2}} \quad ; \quad a, \gamma > 0$$

$$\frac{\partial S}{\partial \beta} = \text{const}$$

$$\int \frac{d\theta}{\sqrt{\beta - 2ma\cos\theta}} - \int \frac{\frac{dr}{r^2}}{\sqrt{2m\left[E - \frac{\gamma}{r^4}\right] - \frac{\beta}{r^2}}} = \text{const}$$

$$\theta = \pi - \theta_1$$

$$-\int \frac{d\theta_1}{\sqrt{\beta + 2ma\cos\theta_1}} - \int \frac{\frac{dr}{r^2}}{\sqrt{2m\left[E - \frac{\gamma}{r^4}\right] - \frac{\beta}{r^2}}} = \text{const}$$

For finite motion, must have $\beta < 0$

$E < 0$

$$-\int \frac{d\theta_1}{\sqrt{2ma\cos\theta_1 - |\beta|}} - \int \frac{\frac{dr}{r^2}}{\sqrt{-2m|E| - \frac{2m\gamma}{r^4} + \frac{|\beta|}{r^2}}} = \text{const}$$

$$|\theta_1| < \cos^{-1}\frac{|\beta|}{2ma}$$

$$\frac{|\beta|}{4\gamma m} - \sqrt{\left(\frac{|\beta|}{4\gamma m}\right)^2 - \frac{|E|}{\gamma}} < \frac{1}{r^2} < \frac{|\beta|}{4\gamma m} + \sqrt{\left(\frac{|\beta|}{4\gamma m}\right)^2 - \frac{|E|}{\gamma}}$$

$$\frac{2}{\sqrt{2ma-|\beta|}}\, F\left(\frac{\theta_1}{2}, \frac{4ma}{2ma-|\beta|}\right) - \frac{1}{\sqrt{2m\gamma}}\, \frac{F\left(\frac{\phi}{2}, \frac{2\sqrt{1-\frac{16|E|m^2\gamma}{\beta^2}}}{1-\sqrt{1-\frac{16|E|m^2\gamma}{\beta^2}}}\right)}{\sqrt{1-\sqrt{1-\frac{16|E|m^2\gamma}{\beta^2}}}}$$

$= \text{const},$

where

$$\phi = \cos^{-1}\left[\frac{\frac{1}{r^2} - \frac{|\beta|}{4\gamma m}}{\sqrt{\left(\frac{|\beta|}{4\gamma m}\right)^2 - \frac{|E|}{\gamma}}}\right]$$

Consider $\frac{|E|}{\gamma} = \left(\frac{|\beta|}{4\gamma m}\right)^2$, $\frac{1}{r} = \frac{|\beta|}{4\gamma m}$ – circular motion *

$$S = \gamma\left(\frac{\beta}{4\gamma m}\right)^2 t + \int d\theta \sqrt{\beta - 2ma\cos\theta}$$

$$\frac{\partial S}{\partial \beta} = \text{const} = \frac{\beta}{8\gamma m^2} t + \frac{1}{2}\int \frac{d\theta}{\sqrt{\beta - 2ma\cos\theta}}$$

$\theta_1 = \pi - \theta$

$$= -\frac{|\beta|}{8\gamma m^2} t - \frac{1}{2}\int \frac{d\theta_1}{\sqrt{-|\beta| + 2ma\cos\theta_1}}$$

$$= -\frac{|\beta|}{8\gamma m^2} t - \frac{F\left(\frac{\theta_1}{2}, \frac{4am}{2am - |\beta|}\right)}{\sqrt{2am - |\beta|}}$$

Initial condition : $t = 0$, $\theta_1 = 0$ $(\theta = \pi) \Rightarrow \text{const} = 0$

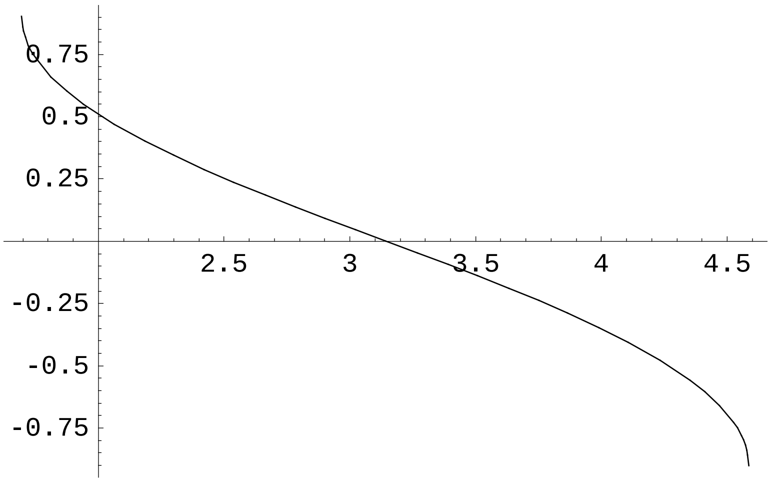
Initial velocity determines β, E, and r

Compare with Add2

constants = $\{a \to 1,\ m \to 1,\ \text{Abs}\beta \to \frac{1}{4}\}$;

t = $\sqrt{\frac{1}{-\text{Abs}\beta + 2\,a\,m}}\ \text{EllipticF}\left[\frac{\pi - \theta}{2}, \frac{4\,a\,m}{-\text{Abs}\beta + 2\,a\,m}\right]$;

Plot$\left[t\ /.\ \text{constants},\ \left\{\theta,\ \pi - \text{ArcCos}\left[\frac{\text{Abs}\beta}{2\,m\,a}\right]\ /.\ \text{constants},\ \pi + \text{ArcCos}\left[\frac{\text{Abs}\beta}{2\,m\,a}\right]\ /.\ \text{constants}\right\}\right]$



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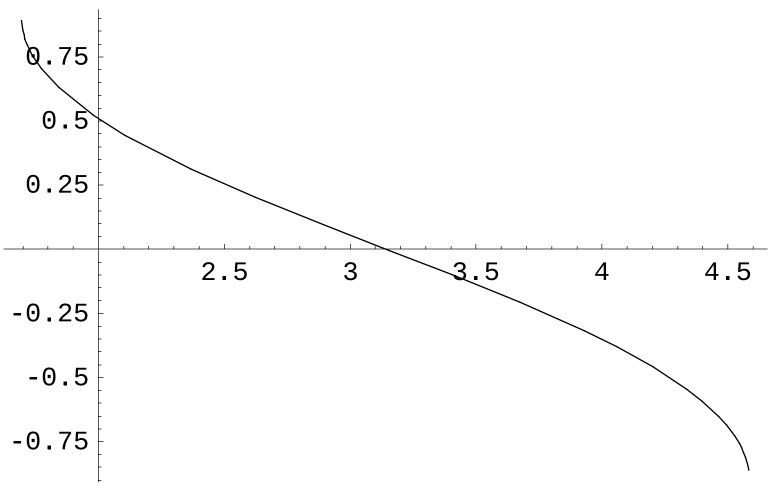
Plot$[t\ /.\ \text{constants},\ \{\theta,\ 0,\ 2\,\pi\}]$



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