11.10

$$\dot{f} = \cancel{\frac{\partial f}{\partial t}}^{\,0} + \{H, f\} \quad \text{see LL (42.1-42.2)}$$

$$\ddot{f} = \{H, \dot{f}\} = \{H, \{H, f\}\}, \text{ etc.}$$

$$f(p(t), q(t)) = f(p(0), q(0)) + \dot{f}\frac{t}{1!} + \ddot{f}\frac{t^2}{2!} + \dots$$

$$= f(p(0), q(0)) + \{H, f\}\frac{t}{1!} + \{H, \{H, f\}\}\frac{t^2}{2!} + \dots$$

(a1) $f = \vec{p}, \quad H = \frac{\vec{p}^2}{2m} - \vec{F}\cdot\vec{r}$

$$\{H, \vec{p}\} \overset{LL(42.12)}{=} -\frac{\partial H}{\partial \vec{r}} = \vec{F}, \quad \{H, \vec{F}\} = 0$$

$$\vec{p} = \vec{p}(0) + \vec{F}t$$

(a3) $f = p^2, \quad \{H, f\} = \{-\vec{F}\cdot\vec{r}, p^2\} = 2\vec{F}\cdot\vec{p}$

$$\{H, \{H, f\}\} = \vec{F}\cdot\{-\vec{F}\cdot\vec{r}, \vec{p}\} = 2F^2$$

$$p^2 = p^2(0) + 2\vec{F}\cdot\vec{p}(0)t + F^2t^2$$

(b) $f = x, \quad H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$

$$\{H, x\} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \{H, \{H, x\}\} = -\omega^2 x$$

$$\{H, \{H, \{H, x\}\}\} = -\frac{\omega^2 p}{m}, \quad \{H, \{H, \{H, \{H, x\}\}\}\} = \omega^4 x$$

11.10 continued

$$x(t) = x(0) + \frac{p(0)}{m}\left(t - \frac{\omega^2 t^3}{3!} + \ldots\right)$$
$$- \omega^2 x(0)\left(\frac{t^2}{2!} - \frac{\omega^2 t^4}{4!} + \ldots\right)$$
$$= x(0) + \frac{p(0)}{m\omega}\left(\omega t - \frac{(\omega t)^3}{3!} + \ldots\right)$$
$$- x(0)\left(\frac{(\omega t)^2}{2!} - \frac{(\omega t)^4}{4!} + \ldots\right)$$
$$= x(0) - x(0)\left[1 - \cos\omega t\right]$$
$$+ \frac{p(0)}{m\omega}\sin\omega t$$
$$= x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

11.9

$$\vec{v} = \frac{1}{m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)$$
$$\{v_k, v_j\} = -\frac{e}{cm^2}\left[\{P_k, A_j\} + \{A_k, P_j\}\right]$$
$$= -\frac{e}{cm^2}\left[\{P_k, A_j\} \cdot \{P_j, A_k\}\right]$$

LL (42.12)
$$= -\frac{e}{cm^2}\left(\frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j}\right) = -\frac{e}{cm^2}\epsilon_{ikj} H_i$$

11.13

$$F = \frac{1}{2} m \omega q^2 \cot Q$$

$$P = -\frac{\partial F}{\partial Q} = \frac{1}{2} m \omega q^2 \csc^2 Q = \frac{1}{2} m \omega \left(\frac{q}{\sin Q}\right)^2$$

$$q = \sin Q \sqrt{2 \frac{P}{m\omega}} \qquad (1)$$

$$p = \frac{\partial F}{\partial q} = m \omega q \cot Q = m\omega \frac{q}{\sin Q} \cos Q$$

$$\overset{\text{using}(1)}{=} m\omega \sqrt{2 \frac{P}{m\omega}} \cos Q \qquad (2)$$

$$\frac{\partial}{\partial \omega} F = \frac{1}{2} m q^2 \cot Q \overset{\text{using}(1)}{=} \frac{1}{2} \frac{P \sin 2Q}{\omega} \qquad (3)$$

$$H' = H + \frac{\partial F}{\partial t} = H + \frac{\partial F}{\partial \omega} \dot{\omega}$$

$$H = \frac{m\omega^2 q^2}{2} + \frac{p^2}{2m} \overset{\text{using (1) and (2)}}{=} \omega P \qquad (4)$$

$$H' = \omega P + \frac{1}{2} \frac{P \sin 2Q}{\omega} \dot{\omega}, \text{ using (4) and (3)}$$

$$\dot{Q} = \frac{\partial H'}{\partial P} = \omega + \frac{1}{2} \frac{\sin 2Q}{\omega} \dot{\omega}$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = -\frac{P \cos 2Q}{\omega} \dot{\omega}$$

11.19

$$q_1 = Q_1 \cos\lambda + \frac{P_2}{m\omega}\sin\lambda$$

$$Q_1 = \frac{q_1 - \frac{P_2}{m\omega}\sin\lambda}{\cos\lambda} \qquad (1)$$

$$p_2 = -m\omega Q_1 \sin\lambda + P_2\cos\lambda$$

$$P_2 = \frac{p_2 + \omega m \sin\lambda\, Q_1}{\cos\lambda} \qquad (2)$$

Substidute (2) in (1), solve for Q_1 and subsdidute back indo (2)

$$Q_1 = q_1\cos\lambda - \frac{p_2}{m\omega}\sin\lambda$$

$$P_2 = p_2\cos\lambda + m\omega q_1 \sin\lambda$$

Similarly

$$Q_2 = q_2\cos\lambda - \frac{p_1}{m\omega}\sin\lambda$$

$$P_1 = p_1\cos\lambda + m\omega q_2\sin\lambda$$

Check

$$\{Q_1, Q_2\} = \frac{\partial Q_1}{\partial p_1}\frac{\partial Q_2}{\partial q_1} - \frac{\partial Q_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1}$$

$$+ \frac{\partial Q_1}{\partial p_2}\frac{\partial Q_2}{\partial q_2} - \frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} = 0$$

11.19 continued

Also $\{Q_1, P_2\} = 0$, $\{Q_1, P_1\} = 1$, etc.

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{m\omega^2(q_1^2 + q_2^2)}{2} = \frac{P_1^2 + P_2^2}{2m} + \frac{m\omega^2(Q_1^2 + Q_2^2)}{2}$$

For $Q_2 = P_2 = 0$

$$Q_1 = A\sin(\omega t + \phi), \quad P_1 = m\omega A\cos(\omega t + \phi)$$

$$q_1 = Q_1 \cos\lambda$$
$$= A\cos\lambda \sin(\omega t + \phi)$$

$$q_2 = \frac{P_1}{m\omega}\sin\lambda$$
$$= A\sin\lambda\cos(\omega t + \phi)$$

$$\frac{q_1^2}{(A\cos\lambda)^2} + \frac{q_2^2}{(A\sin\lambda)^2} = 1 \quad : \text{ellipse}$$

<u>11.25</u>

$$H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + e\phi$$

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} f(\vec{r},t)$$

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

$$\vec{P} \to \vec{P}' \ \ \frac{e}{c}\vec{\nabla} f(\vec{r},t), \quad \vec{r} \to \vec{r}'$$

$$H \to H' = \frac{1}{2m}\left(\vec{P}' - \frac{e}{c}\vec{A}'\right)^2 + e\phi'$$

$$= \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + e\phi - \frac{e}{c}\frac{\partial f}{\partial t}$$

$$= H - \frac{e}{c}\frac{\partial f}{\partial t}$$

$$-\frac{e}{c}\frac{\partial f}{\partial t} = \frac{\partial \Phi}{\partial t}, \quad \Phi = -\frac{e}{c} f + \Delta\Phi(\vec{r},\vec{P}')$$

$$\cancel{\vec{P}} = \vec{\nabla}\Phi = -\frac{e}{c}\vec{\nabla} f + \vec{\nabla}\Delta\Phi(\vec{r},\vec{P}')$$

$$= \cancel{\vec{P}} - \vec{P}' + \vec{\nabla}\Delta\Phi(\vec{r},\vec{P}')$$

$$\Delta\Phi(\vec{r},\vec{P}') = \vec{P}'\cdot\vec{r}$$

$$\Phi = -\frac{e}{c} f + \vec{P}'\cdot\vec{r}$$

11.26

$$\mathcal{L}' = \mathcal{L} + \frac{df}{dt} = \mathcal{L} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q}$$

$$H' = \frac{\partial \mathcal{L}'}{\partial \dot{q}}\dot{q} - \mathcal{L}' = H - \frac{\partial f}{\partial t} = H + \frac{\partial \Phi}{\partial t}$$

$$\frac{\partial \Phi}{\partial t} = -\frac{\partial f}{\partial t}, \quad \Phi = -f + \Delta\Phi(q, P) \quad (1)$$

$$\dot{P} = \frac{\partial}{\partial q}\mathcal{L}' = \frac{\partial}{\partial q}\mathcal{L} + \frac{\partial}{\partial q}\dot{f} = \dot{p} + \frac{\partial}{\partial q}\dot{f}$$

$$P = p + \frac{\partial}{\partial q}f \overset{\text{using}(1)}{=} p - \frac{\partial \Phi}{\partial q} + \frac{\partial \Delta\Phi(q, P)}{\partial q}$$

$$= \cancel{p} - \cancel{p} + \frac{\partial \Delta\Phi(q, P)}{\partial q} \quad (2)$$

$$\Delta\Phi(q, P) = Pq$$

$$\Phi = -f + Pq$$

11.27

$-S$ is the generating function (see end of LL §45)

(b) $H = \frac{p^2}{2m} - Fx, \quad \mathcal{L} = \frac{p^2}{2m} + Fx \qquad (0)$

$$\dot{p} = F, \quad p = P_1 + F(t - T_1) \qquad (1)$$

$$\dot{x} = \frac{p}{m}, \quad x = \frac{F}{m}\left(\frac{t^2}{2} - tT_1\right) + \frac{P_1}{m}t + C$$

$$t = T_2, \quad x = X_2$$
$$t = T_1, \quad x = X_1$$

$$x = X_1 + \frac{F}{2m}(t - T_1)^2 + \frac{P_1}{m}(t - T_1) \qquad (2)$$

$$P_1 = \frac{X_2 - X_1}{T_2 - T_1}m - \frac{1}{2}F(T_2 - T_1) \qquad (3)$$

$$S = \int_{T_1}^{T_2} \mathcal{L}\,dt \overset{\text{using (0), (1), (2), and (3)}}{=}$$

$$= \frac{F}{2}(X_2 + X_1)(T_2 - T_1) + \frac{1}{2m}\frac{(X_2 - X_1)^2}{T_2 - T_1} - \frac{1}{24m}F^2(T_2 - T_1)^3$$

$$F_2(q, Q) = -\frac{F}{2}\tau(q + Q) - \frac{m}{2\tau}(Q - q)^2 + \frac{F^2\tau^3}{24m}$$

II. 27 (continued)

(c) $$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \quad \mathcal{L} = \frac{p^2}{2m} - \frac{m\omega^2 x^2}{2} \qquad (0)$$

$$x = \frac{-X_1 \sin[\omega(t-T_2)] + X_1 \sin[\omega(t-T_1)]}{\sin[\omega(T_2-T_1)]} \qquad (1)$$

$$p = m\dot{x} = m\omega \, \frac{X_2 \cos[\omega(t-T_1)] - X_1 \cos[\omega(t-T_2)]}{\sin[\omega(T_2-T_1)]} \qquad (2)$$

$$S = \int_{T_1}^{T_2} \mathcal{L}\, dt$$

$$= -\frac{m\omega X_2 X_1}{\sin[(T_2-T_1)\omega]} + \frac{\omega m (X_2^2 + X_1^2)\cos[(T_2-T_1)\omega]}{2\sin[(T_2-T_1)\omega]}$$

$$F_\tau(q,Q) = \frac{m\omega q Q}{\sin \omega\tau} - \frac{m\omega(q^2+Q^2)\cos\omega\tau}{2\sin\omega\tau}$$

See also DL Ch. 2