10.3

$$\mathcal{L} = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 - \alpha x^3 + \beta x \dot{x}^2$$

$$p = \frac{\partial}{\partial \dot{x}}\mathcal{L} = (2\beta x + 1)\dot{x}$$

$$\dot{x} = \frac{p}{2\beta x + 1}$$

$$H = \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{x} - \mathcal{L} = \alpha x^3 + \tfrac{1}{2}\omega^2 x^2 + \tfrac{1}{2}(2\beta x + 1)\dot{x}^2$$

$$= \frac{1}{2}\frac{p^2}{2\beta x + 1} + \alpha x^3 + \tfrac{1}{2}\omega^2 x^2$$

For small oscillations, $|\alpha x| \ll \omega^2$ and $|\beta x| \ll 1$,

$$H \approx \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 x^2 + \underbrace{\alpha x^3 - \beta x p^2 + \ldots}_{\delta H}$$

$$\mathcal{L} \approx \tfrac{1}{2}p^2 - \tfrac{1}{2}\omega^2 x^2 \underbrace{- \alpha x^3 + \beta x p^2 - \ldots}_{\delta \mathcal{L}}$$

$\delta H = -\delta \mathcal{L}$ (see LL § 40)

10.4

$$E = H = \left(\tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 x^2\right) + \lambda\left(\tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 x^2\right)^2$$

$$\tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 x^2 = \frac{\sqrt{4\lambda E + 1} - 1}{2\lambda} \quad \left\} \begin{array}{l}\text{solving}\\ \text{quadratic}\\ \text{equation}\end{array}\right.$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\omega_0^2 x - \omega_0^2 x\, 2\lambda\left(\tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 x^2\right)$$

$$= -\omega_0^2 x \sqrt{1 + 4\lambda E} = \dot{p} \qquad (1)$$

$$\dot{x} = \frac{\partial H}{\partial p} = p + 2\lambda\left(\tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 x^2\right)$$

$$= p\sqrt{1 + 4\lambda E} = \dot{x} \qquad (2)$$

From (1) and (2)

$$x = a\cos(\omega t + \phi), \quad p = -\omega_0 a \sin(\omega t + \phi)$$

where

$$\omega = \omega_0 (1 + 4\lambda E)^{1/2}, \quad E = \tfrac{1}{2}\omega_0^2 a^2 + \lambda\left(\omega_0^2 a^2 \tfrac{1}{2}\right)^2$$

$$= \omega_0 (1 + 2\lambda E), \quad E = \tfrac{1}{2}\omega_0^2 a^2$$

10.5

$\vec{p}_0 = p_{x0}\hat{u}_x + p_{\perp 0}\hat{u}_{\perp 0}$, choose $\hat{u}_y = \hat{u}_\perp$

Motion 2D, in (x, y) plane

$$n = ax \, , \quad p = \sqrt{p_x^2 + p_y^2} \, , \quad H = \frac{cp}{n}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{c p_x}{np} \, , \quad dt = \frac{np}{c p_x} dx \qquad (1)$$

$$\dot{y} = \frac{dy}{dt} = \frac{\partial H}{\partial p_y} = \frac{c p_y}{np} \, , \quad dy = \frac{p_y}{p_x} dx \qquad (2)$$

$$\dot{p}_x = \frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial n}\frac{dn}{dx} = \frac{acp}{n^2} \, , \text{ and using (1),}$$

$$\frac{dp_x \, p_x}{(p_x^2 + p_y^2)} = \frac{dx}{x} \qquad (3)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \, , \quad p_y = P = \text{const} \qquad (4)$$

From (3) and (4), by integration,

$$p_x^2 + P^2 = \left(\frac{x}{C}\right)^2 \qquad (5)$$

From (2) and (5)

$$\int dy = \int \frac{dx}{\sqrt{\left(\frac{x}{C}\right)^2 - P^2}} \, , \quad x = CP\cosh\left(\frac{y + B}{CP}\right)$$

$$x = c_1 \cosh\left(\frac{y}{c_1} + c_2\right)$$