Hopf's Bifurcation Theorem and the Center Theorem of Liapunov with Resonance Cases

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1. INTRODUCTION

In recent years numerous papers have dealt with the bifurcation of periodic orbits from an equilibrium point. The starting point for most investigations is the Liapunov Center Theorem [8] or the Hopf Bifurcation Theorem [6]. Local results concerning these theorems were published among others by Chafee [2] who investigated in detail the structure of the periodic orbits in the vicinity of an equilibrium point and by Henrard [4], Schmidt and Sweet [10] who looked at resonance cases in the Liapunov Center Theorem. Global results on the bifurcation of periodic orbits were obtained by Alexander and Yorke [1]. They showed in their paper that Liapunov's Center Theorem can be derived as a consequence of Hopf's Bifurcation Theorem.

Based on a suggestion by J. A. Yorke we show on a local level that Liapunov's theorem can be derived from the one of Hopf. For this we provide an analytic proof of Hopf's theorem based on the method used in [9], which is a special case of the alternative method, which is described in [3]. Our proof is general enough to include the Center Theorem as a corollary, and in addition it is simple enough to allow us to discuss some exceptional cases of the Hopf Bifurcation Theorem.

2. THE HOPF BIFURCATION THEOREM

We consider the n-dimensional autonomous system of differential equations given by

\[ \dot{x} = F(x, \mu) \]  

which depends on the real parameter \( \mu \). Let \( F(x, \mu) \) be twice differentiable in both variables and assume that (1) possesses an analytic family \( x = x(\mu) \) of equilibrium points that is \( F(x(\mu), \mu) = 0 \). Without loss of generality we assume
that this family is given by \( x = 0 \), that is, \( F(0, \mu) = 0 \). We suppose that for a certain value of \( \mu \), say \( \mu = 0 \), the matrix \( F_x(0, \mu) \) has two purely imaginary eigenvalues \( \pm i\beta \) and no other eigenvalue of \( F_x(0, 0) \) is an integral multiple of \( i\beta \). If \( \alpha(\mu) + i\beta(\mu) \) is the continuation of the eigenvalue \( i\beta \) then we assume that \( \alpha'(0) \neq 0 \).

**Theorem (Hopf).** Under the above conditions there exists differentiable functions \( \mu = \mu(\epsilon) \) and \( T = T(\epsilon) \) depending on a parameter \( \epsilon \) with \( \mu(0) \) and \( T(0) = 2\pi\beta^{-1} \) such that there are nonconstant periodic solutions \( x(t, \epsilon) \) of (1) with period \( T(\epsilon) \) which collapse into the origin as \( \epsilon \to 0 \).

**Proof.** With the help of a linear change of coordinates and by changing the independent variable to \( \tau = \beta(\mu) t \) we can bring equation (1) into the following form

\[
\begin{align*}
\dot{x}_1 &= (\alpha(\mu) + i) x_1 + f_1(x_1, x_2, y, \mu) \\
\dot{x}_2 &= (\alpha(\mu) - i) x_2 + f_2(x_1, x_2, y, \mu) \\
\dot{y} &= B(\mu) y + g(x_1, x_2, y, \mu)
\end{align*}
\]

(2)

\( y \) is a real \((n - 2)\) dimensional vector. \( B(\mu) = B_0 + O(\mu) \) is a real \( n - 2 \) square matrix not necessarily in normal form and by assumption no imaginary integer is an eigenvalue of \( B_0 \). Real solutions are only given when \( x_1 = x_2 \) and therefore the following reality condition holds \( f_1(x_1, x_2, y, \mu) = f_2(x_2, x_1, y, \mu) \). Furthermore, the functions \( f_1, f_2 \) and \( g \) vanish with their first partial derivatives for \( x_1 = x_2 = 0, y = 0 \).

We introduce the scale factor \( \epsilon \) by

\[
\mu = \epsilon \mu_1, x_j = \epsilon^2 \xi_j, j = 1, 2, y = \epsilon^2 \eta
\]

into system (2) and obtain

\[
\begin{align*}
\dot{\xi}_1 &= i \xi_1 + \epsilon \mu_1 \alpha'(0) \xi_1 + O(\epsilon^2) \\
\dot{\xi}_2 &= -i \xi_2 + \epsilon \mu_1 \alpha'(0) \xi_2 + O(\epsilon^2) \\
\eta &= B_0 \eta + O(\epsilon)
\end{align*}
\]

In our search for periodic solutions with period near \( 2\pi \) we see that for \( \epsilon = 0 \) we must have \( \eta = 0 \) whereas \( \xi_1 \) and \( \xi_2 \) can have arbitrary initial conditions. Due to the autonomous character of the system and due to our scaling we can restrict ourselves to the initial condition \( \xi_1(0) = \xi_2(0) = 1 \) and therefore get for \( \epsilon = 0 \) the \( 2\pi \) periodic solution \( \xi_1 = e^{it}, \xi_2 = e^{-it}, \eta = 0 \). For \( \epsilon \neq 0 \) we can expect to find nearby periodic solutions of period \( T = 2\pi(1 + \epsilon \delta) \) if we can satisfy the periodicity condition

\[
\begin{align*}
\xi_j(T) - \xi_j(0) &= 0 \quad j = 1, 2 \\
\eta(T) - \eta(0) &= 0.
\end{align*}
\]
With $\xi_1(0) = \xi_2(0) = 1$ and $\gamma(0) = \gamma_0$ we arrive at the following bifurcation equations

$$
\Gamma_1 = \frac{1}{2\pi \epsilon} (\xi_1(T) - 1) = i\delta + a'(0) \mu_1 + O(\epsilon)
$$

$$
\Gamma_2 = \frac{1}{2\pi \epsilon} (\xi_2(T) - 1) = -i\delta + a'(0) \mu_1 + O(\epsilon)
$$

$$
\Gamma = \eta(T) - \eta_0 = (e^{\pi R_0} - I) \eta_0 + O(\epsilon).
$$

For $\epsilon = 0$ this system has the unique solution $\delta = 0$, $\mu = 0$ and $\gamma_0 = 0$. Since the Jacobian $\frac{\partial (\Gamma_1, \Gamma_2, \Gamma)}{\partial (\delta, \mu, \gamma_0)}$ has rank $n$ for $\epsilon = 0$, the bifurcation equations can be solved by the implicit function theorem for $|\epsilon| \neq 0$ but sufficiently small to give $\delta$, $\mu$ and $\gamma_0$ as functions of $\epsilon$.

Remarks. Our assumptions for Hopf's theorem are slightly less restrictive than they are usually stated as we do not require that the other eigenvalues are non-imaginary as long as the resonance cases are excluded.

The function $F(x, \mu)$ can be $C^r$ or analytic and it follows that the family of periodic orbits depends $C^{r-1}$ or analytically on the parameter $\epsilon$. In the remaining sections we will assume for convenience that $F(x, \mu)$ is analytic or at least as often differentiable as required.

In order to justify some of the scalings which are used later on it may be helpful to point out that in the variables of system (2) the periodic orbits occurs for $x = O(\epsilon^3)$, $y = O(\epsilon^4)$, $\mu = O(\epsilon^3)$ and $T = 2\pi + O(\epsilon^2)$.

Finally, the proof would have been similar if we had used the real normal form of (2), but the complex form turns out to be more convenient when we discuss the case $a'(0) = 0$ in section 5 and resonance cases in section 6.

### 3.3. The Liapunov Center Theorem

**Theorem (Liapunov).** Consider the system

$$
\dot{x} = Ax + f(x)
$$

where $f$ is a smooth function which vanishes along with its first partial derivatives at $x = 0$. Assume that the system admits a first integral of the form $I(x) = \frac{1}{2} x^T S x + \cdots$ where $S = S^T$ and $\det S \neq 0$. Let $A$ have eigenvalues $-i\beta$, $\lambda_1, \ldots, \lambda_n$, $\beta \neq 0$. Then if $\lambda_j |i\beta| \neq \text{integer for } j = 3, \ldots, n$ the above system has a one parameter family of periodic solutions emanating from the origin starting with period $2\pi/\beta$.

**Proof.** As announced in the introduction we will show that this theorem is a consequence of the Hopf bifurcation theorem. To this end we consider the modified system

$$
\dot{x} = Ax + f(x) + \mu \text{ grad } I(x)
$$
and we will show that all conditions of Hopf's theorem are met and that the nonstationary periodic orbits can only occur for \( \mu = 0 \).

The second part is easily done by evaluating \( \frac{dI}{dt} \) along solutions of (4), which gives

\[
\frac{dI}{dt} = \langle \nabla I(x), Ax + f(x) + \mu \nabla I(x) \rangle
= \mu |\nabla I(x)|^2.
\]

The second equality holds because \( I(x) \) is an integral for (3). Therefore, \( 1/\mu \frac{dI}{dt} \) is monotonically increasing unless \( \nabla I(x(t)) = 0 \), which gives \( x(t) = 0 \) that is a stationary point, because \( \nabla I(x) = Sx + \cdots = 0 \) has \( x = 0 \) as an isolated solution.

In order to apply the theorem of the previous section we only have to verify the condition concerning the real part of the eigenvalue near \( i\beta \). Again through a linear change we will bring the linear part of the system (3) into a normal form. We assume that this has been done already and the matrix \( A \) has thus the following real form

\[
A = \begin{pmatrix}
0 & \beta & 0 \\
-\beta & 0 & 0 \\
0 & 0 & \bar{A}
\end{pmatrix}
\]

where \( \bar{A} \) is a real \( n - 2 \) square matrix. It follows from \( ATS + SA = 0 \) that the matrix \( S \) in the integral has the form

\[
S = \begin{pmatrix}
a & 0 & 0 \\
0 & a & 0 \\
0 & 0 & \bar{S}
\end{pmatrix}
\]

with \( a \neq 0 \) since \( \det S \neq 0 \).

From the matrix

\[
A + \mu S = \begin{pmatrix}
\mu a & \beta & 0 \\
-\beta & \mu a & 0 \\
0 & 0 & \bar{A} + \mu \bar{S}
\end{pmatrix}
\]

it follows at once that the eigenvalue near \( i\beta \) has real part \( \alpha(\mu) = a\mu \) and therefore \( \alpha'(0) = a \neq 0 \).

4. Stability

The stability of the periodic solution depends on higher order terms in the differential equation (2). In order to derive the stability criterion we will first use
some nonlinear transformation to eliminate as many higher order terms as possible. Without this the stability condition would result in page long formulas. This elimination procedure can be accomplished most easily with the help of the Lie transform. It is still a tedious job but it is suitable for implementing on a computer and therefore it will not be too long until computer programs are available which perform the required algebraic manipulations automatically. (For 2 dimensional systems see [5]). For this reason we will state our results in terms of the coefficients of the normalized system. The normal form can also be learned from [12].

For the convenience of the reader we will outline the method of the Lie transform, as described in [5]. The change of variables \( z = z(\xi, \epsilon) \) which transforms the system

\[
\dot{z} = Z_*(z, \epsilon) = \sum_{0}^{N} \frac{\epsilon^i}{i!} Z_i^0(z) + O(\epsilon^{N+1})
\]

into

\[
\dot{\xi} = Z^*(\xi, \epsilon) = \sum_{0}^{N} \frac{\epsilon^i}{i!} Z_0^i(\xi) + O(\epsilon^{N+1})
\]

is constructed as the solution of the system of differential equations

\[
\frac{d z}{d \epsilon} = W(z, \epsilon) = \sum_{0}^{N-1} \frac{\epsilon^i}{i!} W_{i+1}(z)
\]

with the initial condition \( z(0) = \xi \). The transformation proceeds via a double indexed array of functions \( \{Z^i_j\} \) which agree with the previous definition when \( i \) or \( j \) is zero. These functions are conveniently displayed in the triangular array

\[
\begin{array}{c}
 Z^0_0 \\
 Z^0_1 \rightarrow \nearrow Z^1_0 \\
 Z^0_2 \rightarrow \nearrow \rightarrow \nearrow Z^2_0 \\
 \downarrow \downarrow \downarrow \downarrow \downarrow
 \end{array}
\]

and are defined recursively by

\[
Z^i_j = Z^i_{j+1} + \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} L_{k+1} Z^{i-1}_{j-k}
\]
where the Lie derivative is defined by

$$L_k Z = \frac{\partial Z}{\partial z} W_k - \frac{\partial W_k}{\partial z} Z.$$  

Terms that can be eliminated in $Z_0^k$ by the proper choice of $W_k$ lie in the range of the Lie derivative $L_k Z_0^0$.

We will be satisfied to normalize second and third order terms of (2). Therefore we write $f$ and $g$ as the sum of homogeneous polynomials of degree two and three plus higher order terms and introduce the scale factor $\epsilon$ by

$$x \rightarrow \epsilon x, \ y \rightarrow \epsilon y \quad \text{and} \quad \mu \rightarrow \epsilon \mu.$$  

We consider the differential equations for $x$ first. With the notation $x^\alpha = x_1^\alpha x_2^\alpha$, $|\alpha| = \alpha_1 + \alpha_2$, etc., it reads

$$\dot{y} \rightarrow B_0 y + \epsilon \sum_{|\alpha| + |\beta| = 2} a_{\alpha\beta} x^\alpha y^\beta + \cdots.$$  

In a preliminary step we show that all quadratic terms involving only $x_1$ and $x_2$ can be eliminated. Since

$$L_w Z_0^0 = \frac{\partial Z_0^0}{\partial z} W - \frac{\partial W}{\partial z} Z_0^0$$

with

$$z = \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \quad \text{and} \quad Z_0^0 = \begin{pmatrix} ix_1 \\ -ix_2 \\ B_0 y \end{pmatrix}$$

is a linear mapping of the space of homogeneous polynomials of fixed degree into itself, it will be sufficient to consider the three cases separately which arise from $|\alpha| = \alpha_1 + \alpha_2 = 2$. With $c$ a $(n - 2)$-dimensional complex column vector and

$$W = \begin{pmatrix} 0 \\ 0 \\ c x^\alpha \end{pmatrix}$$

we get

$$L_w Z_0^0 = \begin{pmatrix} 0 \\ 0 \\ (B_0 - i(x_1 - \alpha_2 I) c x^\alpha) \end{pmatrix}$$

Since 0 and $\pm 2i$ are not eigenvalues of $B_0$ by assumption the above quadratic terms can be eliminated. This allows us to scale an additional time by $y \rightarrow \epsilon y$.

The first equation of (2) reads now

$$\dot{x}_1 = ix_1 + \epsilon \sum_{|\alpha|=2} c_\alpha x^\alpha + \epsilon^2 \left( a'(0) \mu x_1 + (x_1 p + x_2 q) y + \sum_{|\alpha|=3} c_\alpha x^\alpha \right) + O(\epsilon^3)$$
where $p$ and $q$ are $(n - 2)$ dimensional row vectors. Let $w(x, y)$ be the first component of $W$, then the corresponding Lie derivative reads

$$L_w Z_0^0 = i \left( w - x_1 \frac{\partial w}{\partial x_1} + x_2 \frac{\partial w}{\partial x_2} \right) - \frac{\partial w}{\partial y} B_0 y.$$ 

It follows that all quadratic terms can be eliminated. On the other hand when $w = x_1 x_2^2$ with $\alpha_1 + \alpha_2 = 3$ we get

$$L_w Z_0^0 = i(1 - \alpha_1 + \alpha_2) x_1 x_2^2$$

which is zero when $\alpha_1 = 2, \alpha_2 = 1$. Thus we are left with the following system of differential equations.

$$\begin{align*}
\dot{x}_1 &= i x_1 + e^2 x_1 (a'(0) \mu + Q x_1 x_2) + O(\epsilon^3) \\
\dot{x}_2 &= -i x_2 + e^2 x_2 (a'(0) \mu + \bar{Q} x_1 x_2) + O(\epsilon^3) \\
\dot{y} &= B_0 y + O(\epsilon)
\end{align*}$$

(5)

where $Q = Q_1 + iQ_2$ is in general a complex quantity. For this system we provide the following addition to Hopf's theorem.

**Lemma.** The characteristic multipliers of the periodic orbit are $1, 1 + 4\pi e^2 Q_1 + O(\epsilon^3), \lambda_3 + O(\epsilon), ..., \lambda_n + O(\epsilon)$ where $\lambda_3, ..., \lambda_n$ are the eigenvalues of $e^{2\pi B_0}$.

**Proof.** For the periodic orbit of system (5) we will derive the bifurcation equation again as they provide at the same time the characteristic multipliers. We are looking for a periodic orbit of period $T = 2\pi (1 + \epsilon Q_2)$ with initial conditions $x_1(0) = \xi_1, x_2(0) = \xi_2, y(0) = \eta$. Therefore, we have to solve the equations

$$\begin{align*}
\Gamma_1 &= \frac{1}{2\pi e^2} (x_1(T) - \xi_1) = \xi_1 (i\delta + a'(0) \mu + Q \xi_1 \xi_2) + O(\epsilon) = 0 \\
\Gamma_2 &= \frac{1}{2\pi e^2} (x_2(T) - \xi_2) = \xi_2 (-i\delta + a'(0) \mu + \bar{Q} \xi_1 \xi_2) + O(\epsilon) = 0 \\
\Gamma &= y(T) - \eta = (e^{2\pi B_0} - I) \eta + O(\epsilon) = 0.
\end{align*}$$

(6)

For $\epsilon = 0$ this system has the solution $\xi_1 = \xi_2$ arbitrary, say $\xi_1 = 1, \eta = 0, \delta = -Q_2, \mu = -Q_1 a'(0)$. Since

$$\left| \frac{\partial (\Gamma_1, \Gamma_2, \Gamma)}{\partial (\delta, \mu, \eta)} \right|_{\delta = 0} = 2ia'(0) \det(e^{2\pi B_0} - I) \neq 0.$$
this solution persists by the implicit function theorem for \( |\epsilon| \neq 0 \) but sufficiently small. The characteristic multipliers of the periodic orbit are the eigenvalues of the matrix \( \phi = \partial(x_1(T), x_2(T), y(T))/\partial(\xi_1, \xi_2, \eta) \). With the help of equation (6) we find the matrix

\[
\phi = \begin{pmatrix}
1 + 2\pi\epsilon^2 Q & 2\pi\epsilon^2 Q & O(\epsilon^3) \\
2\pi\epsilon^2 Q & 1 + 2\pi\epsilon^2 Q & O(\epsilon^3) \\
O(\epsilon) & O(\epsilon) & e^{2nB_0} + O(\epsilon)
\end{pmatrix}
\]

whose eigenvalues are those given in the lemma.

If all eigenvalues of \( B_0 \) have negative real parts and \( a'(0) > 0 \) then the following standard statements are now obvious: If bifurcation occurs above criticality (\( \mu = -Q_1/a'(0) > 0 \)) then the orbit is stable, if bifurcation occurs below criticality (\( \mu = -Q_1/a'(0) < 0 \)) then the orbit is unstable. In the case of the Liapunov center theorem \( Q_1 = 0 \), but in this case it is well known that two characteristic multipliers are always 1.

5. Exceptional Cases for the Hopf Bifurcation Theorem

Our proof of Hopf's theorem is easy enough to enable us to discuss the case when the real part of the eigenvalue does not satisfy the condition \( a'(0) \neq 0 \), but instead \( a''(0) \neq 0 \). More generally we assume that \( a(0) = a'(0) = \cdots = a^{(m-1)}(0) = 0 \) and \( a^{(m)}(0) \neq 0 \) (\( m > 1 \)). This time we scale by \( x \rightarrow \epsilon^m x \), \( y \rightarrow \epsilon^{m+1} y \) and \( \mu \rightarrow \epsilon^m \mu \). After normalization which has been described in the previous section we arrive at a system which looks similar to (5). From now on we will only write down the equation for \( x_1 \) as the equation for \( x_2 \) is conjugate to it and the equation for \( y \) has the form given in (5). For \( x_1 \) we find

\[
x_1 = ix_1 + \epsilon^m x_1(a^{(m)}(0) \mu^m/m! + Qx_1x_2) + O(\epsilon^{m+1}).
\]

To find periodic orbits of period \( T = 2\pi(1 + \epsilon^m \delta) \) with initial conditions \( x_1(0) = \xi_1, x_2(0) = \xi_2, y(0) = \eta \) we have to solve a set of bifurcation equations of which the first one reads

\[
\Gamma_1 = \xi_1(i\delta + a^{(m)}(0) \mu^m/m! + Q\xi_1\xi_2) + O(\epsilon) = 0.
\]

For \( \epsilon = 0 \) with \( \xi_1 = \xi_2 = 1 \) we can always find from this set \( \delta = -Q_2 \) and \( \eta = 0 \), but to find \( \mu \) we have to distinguish between \( m \) even or odd. In case \( m \) is odd we always have one solution \( \mu = (-Q_1 m!/a^{(m)}(0))^{1/m} \). In case \( m \) is even and \( Q_1 a^{(m)}(0) > 0 \) we get no solution, on the other hand, if \( Q_1 a^{(m)}(0) < 0 \) we get two solutions \( \mu = \pm (-Q_1 m!/a^{(m)}(0))^{1/m} \).
When we have a solution for $E = 0$ the Jacobian is
$$
\left| \frac{\partial \Gamma(1, 1, 1)}{\partial (\delta, \mu, \eta)} \right|_{\epsilon = 0} = 2i \frac{d^{(m+1)}(0)}{(m - 1)!} \mu^{m-1} \det(e^{2\pi B_0} - I).
$$
It is different from zero when $Q_1 \neq 0$ and thus by the implicit function theorem
the solution of the bifurcation equations persists for $|\epsilon| \neq 0$ but small. Actually,
when $m$ is odd we do not need the implicit function theorem to find a solution
for $\mu$ even when $Q_1 = 0$. But of course the uniqueness guaranteed by the use of
the implicit function theorem is then not available.

The characteristic multipliers of the periodic orbit are computed as in section 4
and we thus have the following result, which is related to the work done in [2],
except that we use a different hypothesis:

**THEOREM.** Consider system (2) with $a(0) = a'(0) = \cdots = a^{(m-1)}(0) = 0,$
$a^{(m)}(0) \neq 0$ in its normalized form

$$
\begin{align*}
\dot{x}_1 &= (a(\mu) + i)x_1 + Qx_1^2x_2 + \phi_1(x_1, x_2, y, \mu) \\
\dot{x}_2 &= (a(\mu) - i)x_2 + Qx_1x_2^2 + \phi_2(x_1, x_2, y, \mu) \\
\dot{y} &= B(\mu)y + \psi(x_1, x_2, y, \mu)
\end{align*}
$$

where $\phi_1$ and $\phi_2$ do not contain quadratic or cubic terms in $x_1$ and $x_2$ and $\psi$ does not
contain quadratic terms in $x_1$ and $x_2$, that is the scaling

$$
x \to \epsilon^m x, \quad y \to \epsilon^{m+1} y \quad \text{and} \quad \mu \to \epsilon^2 \mu
$$

will give (6). Let $Q = Q_1 + iQ_2$ and assume $Q_1 \neq 0$.

Then if $m$ is odd there exists at least locally a one parameter family of periodic
solutions which collapse into the origin as $\epsilon \to 0$ and the period tends to $2\pi$.

If $m$ is even and $a^{(m)}(0)Q_1 < 0$ there are two such families, one occurs for $\mu(\epsilon) > 0$,
the other for $\mu(\epsilon) < 0$. In case $a^{(m)}(0)Q_1 > 0$ there are no such families.

These families are unique in the sense that there are no other periodic solutions
in the vicinity of the origin of period close to $2\pi$ when $\mu$ is near zero. The characteristic
multipliers of the periodic orbit are $1, 1 + 4\pi e^m Q_1 + O(\epsilon^{m+1})$ and the
eigenvalues of $e^{2\pi B_0} + O(\epsilon)$.

The connection to Chafee's work can be seen if we use more specific assumptions.
They are $Q_1 < 0, a(\mu) > 0$ for $0 < \mu < \mu_0$ and all eigenvalues of $B(\mu)$
have negative real part. We then can use the central manifold theorem (see [7])
to transform (2) such that $g(x_1, x_2, 0, \mu) = 0$. This means that the periodic
orbits can be studied in a two dimensional phase plane $y = 0$ where their
behavior becomes obvious.
These assumptions tell us that for $\mu > 0$ the origin is an unstable spiral point. The spirals tend to an asymptotically orbital stable limit cycle which is predicted by the above theorem. As $\mu \to 0$ the limit cycle shrinks into the origin. This makes the origin a stable spiral point for $\mu = 0$, which is also seen from the Liapunov function $V = |x_1|^2$ for which $\dot{V} = 2Q_1 |y_1|^4 + \text{higher order terms}$ is negative definite.

If $m$ is even then $a(\mu) > 0$ at least initially as $\mu$ goes to negative values and the origin becomes again an unstable spiral point, where the spirals tend to a periodic orbit of the second family. If $m$ is odd then $a(\mu) < 0$ for $\mu < 0$ and the origin remains a stable spiral point with no periodic orbits nearby.

6. Resonance Cases in the Hopf Bifurcation Theorem

In this section we want to discuss what happens to the Hopf bifurcation when a second pair of eigenvalues crosses the imaginary axis at $\mu = 0$ and is there an integral multiple of the first pair. For simplicity we restrict ourselves to a four dimensional system, which we assume to be analytic or sufficiently often differentiable. After a change in the time scale the eigenvalues of the linear system are

$$a(\mu) \pm i \quad \text{and} \quad b(\mu) \pm ic(\mu).$$

Since we will assume $a'(0) \neq 0$ we can reparameterize the system such that $a(\mu) = \mu$ for $\mu$ near zero. In the new parameter we write

$$b(\mu) = b_1 \mu + \cdots,$$
$$c(\mu) = k + c_1 \mu + \cdots \quad k \text{ integer}$$

We will again use complex coordinates $x_1, x_2$ with their conjugate values denoted by $y_1$ and $y_2$. The differential equations for $x_1$ and $x_2$ are

$$\dot{x}_1 = (b(\mu) + ic(\mu)) x_2 + \phi_2(x_1, y_1, x_2, y_2, \mu)$$
$$\dot{x}_2 = (b(\mu) + ic(\mu)) x_1 + \phi_1(x_1, y_1, x_2, y_2, \mu)$$

Since real solutions are given by $x_j = \overline{y}_j$, $j = 1, 2$ the differential equations for $y_1$ and $y_2$ are the conjugate complex of those for $x_1$ and $x_2$ and thus do not have to be written down.

In a preliminary transformation we try to simplify some of the lower order terms in $\phi_1$ and $\phi_2$. We write those terms as homogeneous polynomials starting with degree 2 in the form

$$\sum c_{\alpha_1 \beta_1 \alpha_2 \beta_2} x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2}.$$
We will use the scaling
\[ x_j \to \epsilon x_j, \; y_j \to \epsilon y_j, \; j = 1, 2 \quad \text{and} \quad \mu \to \epsilon^2 \mu \] (8)
in order to apply the Lie transformation. From the corresponding Lie derivative we find that those terms can not eliminated whose exponents satisfy the following relations,
\[ \text{in } \phi_1: \quad \alpha_1 - \beta_1 - 1 + k(x_2 - \beta_2) = 0 \]
\[ \text{in } \phi_2: \quad \alpha_1 - \beta_1 + k(x_2 - \beta_2 - 1) = 0 \] (9)
These conditions necessitate that we separate the low resonance cases (|k| \leq 3) from the others in our discussion.

The case |k| \geq 4. We only normalize first and second order terms in \( \epsilon \). Since periodic solutions of long period near 2\pi as compared to those of shorter period near 2\pi/k are expected near the \( x_2 = y_2 = 0 \) plane we scale an additional time by \( x^2 + \epsilon x_2, \; y^2 + \epsilon y_2 \). The system (7) has then the form
\[ \dot{x}_1 = ix_1 + \epsilon^3 x_1(\mu + A x_1 y_1) + O(\epsilon^3) \]
\[ \dot{x}_2 = i k x_2 + \epsilon^2 x_2((b_1 + i c_1) \mu + C x_1 y_1) + O(\epsilon^{5/2}) \]
where \( A = A_1 + i A_2 \) and \( C = C_1 + i C_2 \) are in general complex quantities. For later on we call \( M = C - \epsilon A_1 \). In our search of periodic orbits of period \( 2\pi(1 + \epsilon^2 \delta) \) and initial conditions \( x_j(0) = \xi_j, \; y_j(0) = \eta_j = \bar{\xi}_j, \; j = 1, 2 \) we have to solve the bifurcation equations
\[ \Gamma_1 = \xi_1(\epsilon \delta + \mu + A \xi_1 \eta_1) + O(\epsilon) \]
\[ \Gamma_2 = \xi_2(ik \delta + (b_1 + ic_1) \mu + C \xi_1 \eta_1) + O(\epsilon^{1/2}) \]
This system of four real equations has for \( \epsilon = 0 \) and for nonzero \( \xi_1 \), say \( \xi_1 = 1 \), the solution \( \xi_2 = 0, \; \delta = -A_2, \; \mu = -A_1 \). Provided that the Jacobian satisfies
\[ \left| \frac{\partial (\Gamma_1, \Gamma_2, \Gamma_1, \Gamma_2)}{\partial (\delta, \mu, \xi_2, \eta_2)} \right|_{\epsilon = 0} = 2i((C_1 - b_1 A_1)^2 + (M - c_1 A_1)^2) \neq 0 \] (10)
we can use the implicit function theorem to find the four quantities of \( \xi_2, \delta, \mu \) as functions of \( \epsilon \). From the matrix \( \partial (\Gamma_1, \Gamma_2, \Gamma_1, \Gamma_2)/\partial (\xi_1, \xi_2, \eta_1, \eta_2) \) we also find that the characteristic multipliers are
\[ 1, 1 + 4\pi \epsilon^2 A_1 + O(\epsilon^{5/2}), \; \exp\{4\pi \epsilon^2(C_1 - b_1 A_1 \pm i(M - c_1 A_1)) + O(\epsilon^{5/2})\} \]
The case \( k = 2 \). The condition (9) shows that two quadratic terms in (7), one with \( y_1 x_2 \) in \( \phi_1 \) the other with \( x_1^2 \) in \( \phi_2 \), can not be eliminated. Due to their
presence we modify our scaling (8) to \( \mu \to \epsilon \mu \) and stop after the normalization of the quadratic terms. The differential equations for \( x_1 \) and \( x_2 \) in their normal form read

\[
\begin{align*}
\dot{x}_1 &= i x_1 + \epsilon (\mu x_1 + E y_1 x_2) + O(\epsilon^2) \\
\dot{x}_2 &= 2i x_2 + \epsilon (\mu (b_1 + ic_1) x_2 + F x_1^2) + O(\epsilon^2).
\end{align*}
\]

The complex numbers \( E \) and \( F \) will be assumed to be nonzero and thus without loss of generality we can write \( E = e^{i\alpha} \) and \( F = e^{i\beta} \).

For \( \epsilon = 0 \) we find the \( 2\pi \) periodic solutions

\[
\begin{align*}
x_1 &= \xi_1 e^{i\theta}, & x_2 &= \xi_2 e^{2i\theta} \\
y_1 &= \eta_1 e^{-i\theta} = \bar{x}_1, & y_2 &= \eta_2 e^{-2i\theta} = \bar{x}_2
\end{align*}
\]

These periodic solutions continue to exist for \( \epsilon \neq 0 \) with period \( 2\pi (1 + \epsilon \delta) \) if the following bifurcation equation can be satisfied.

\[
\begin{align*}
\Gamma_1 &= i \delta \xi_1 + \mu \xi_1 + E \xi_2 \eta_1 + O(\epsilon) = 0 \\
\Gamma_2 &= 2i \delta \xi_2 + \mu (b_1 + ic_1) \xi_2 + F \xi_1^2 + O(\epsilon) = 0.
\end{align*}
\]

For \( \epsilon = 0 \) we are not interested in the solutions with \( \xi_1 = \eta_1 = 0 \) since these would give short periodic orbits. They belong to the eigenvalue \( 2i \) and are already guaranteed by Hopf's theorem. Therefore we set \( \xi_1 = \eta_1 = 1 \) and introduce the notation

\[
E \xi_2 = \sqrt{\rho} \ e^{i\phi} \quad \text{and} \quad E \cdot F = e^{i(\phi + \beta)} = e^{i\gamma}
\]

The bifurcation equations then correspond for \( \epsilon = 0 \) to the following four real equations

\[
\begin{align*}
\mu &= -\sqrt{\rho} \cos \phi \\
\delta &= -\sqrt{\rho} \sin \phi \\
b_1 \rho \cos \phi - \cos(\gamma - \phi) &= 0 \\
2\rho \sin \phi + c_1 \rho \cos \phi - \sin(\gamma - \phi) &= 0.
\end{align*}
\]

For fixed \( \rho \) the third equation has two solutions for \( \phi \) determined by

\[
\begin{align*}
\cos \phi &= \sin \gamma (\sin^2 \gamma + (b_1 \rho - \cos \gamma)^2)^{-1/2} \\
\sin \phi &= (b_1 \rho - \cos \gamma)(\sin^2 \gamma + (b_1 \rho - \cos \gamma)^2)^{-1/2}
\end{align*}
\]

The two solutions for \( \phi \) differ by \( \pi \) depending on the sign of the square root.
The fourth equation in (11) with the above values of $\phi$ gives then the following determining relation for $\rho$

$$2b_1\rho^3 + (c_1 \sin \gamma + (b_1 - 2) \cos \gamma) \rho - 1 = 0 \quad (13)$$

We are only interested in positive roots of this equation. Since each $\rho$ corresponds to two values of $\phi$ there are 0, 2 or 4 families of periodic orbits possible. These families persist for $\epsilon \neq 0$ if the Jacobian evaluated at the above solutions is nonzero. The expression for the Jacobian is

$$=-2i \left[2b_1 \rho + (2 + b_1) \cos(\gamma - 2\phi) + c_1 \sin(\gamma - 2\phi) + \frac{1}{\rho}\right] \quad (14)$$

The case $k = -2$. The normal form of the differential equation (7) is now

$$x_1 = i\xi_1 + \epsilon(\mu \xi_1 + F\eta_1\eta_2) + O(\epsilon^2)$$
$$\dot{x}_2 = -2ix_2 + \epsilon((b_1 + ic_1) \mu x_2 + F\eta_1^2) + O(\epsilon^2).$$

We assume again that $E = e^{i\alpha}$ and $F = e^{i\beta}$ and find the bifurcation equations

$$\Gamma_1 = i\delta \xi_1 + \mu \xi_1 + E\eta_1\eta_2 + O(\epsilon)$$
$$\Gamma_2 = -2i\delta \xi_2 + (b_1 + ic_1) \mu \xi_2 + F\eta_1^2 + O(\epsilon)$$

With the notation $E\xi_2 = \sqrt{\rho} e^{i\phi}$ and $EF = e^{i(\beta - \alpha)} = e^{i\gamma}$ the determining relation for $\phi$ and $\rho$ and the Jacobian turn out to be identical to those of the case $k = 2$, (see (12), (13) and (14)). Therefore the same conclusion as earlier holds and it is possible to have 0, 2 or 4 families of periodic orbits bifurcating from the equilibrium.

The case $k = 3$. As seen from (9) many terms of order $\epsilon^2$ will appear now in the normal form of the differential equations (7) which is

$$x_1 = i\xi_1 + \epsilon^2(\mu \xi_1 + Ax_1^2y_1 + Bx_1x_2y_2 + Ey_1^2x_2) + O(\epsilon^2)$$
$$\dot{x}_2 = 3i\xi_2 + \epsilon^2(\mu(b_1 + ic_1) x_2 + Cx_1y_1x_2 + Dx_2^2y_2 + Fx_1^3) + O(\epsilon^2).$$

We again assume that $E$ and $F$ are nonzero and thus can made to have unit modulus. The bifurcation equations become

$$\Gamma_1 = (i\delta + \mu + A\xi_1\eta_1 + B\xi_2\eta_2) \xi_1 + E\eta_1^2\xi_2 + O(\epsilon) = 0$$
$$\Gamma_2 = (3i\delta + \mu(b_1 + ic_1) + C\xi_1\eta_1 + D\xi_2\eta_2) \xi_2 + F\xi_1^2 + O(\epsilon) = 0.$$
$E\xi_2 = \sqrt{\rho} e^{i\phi}$. The bifurcation equations with $\xi_1 = \eta_1 = 1$ read for $\epsilon = 0$ in real form

\[
\delta + A_2 + B_2\rho + \sqrt{\rho} \sin \phi = 0
\]
\[
\mu + A_1 + B_1\rho + \sqrt{\rho} \cos \phi = 0
\]
\[
(\mu b_1 + C_1 + D_1\rho) \sqrt{\rho} + \cos(\gamma - \phi) = 0
\]
\[
(3\delta + \mu c_1 + C_2 + D_2\rho) \sqrt{\rho} + \sin(\gamma - \phi) = 0.
\]

From the first 3 equations we determine $\phi$ and then together with the last find a determining relation for $\rho$. With the abbreviations

\[
M_0 = C_1 - b_1A_1, \quad M = C_2 - 3A_2 - c_1A_1
\]
\[
N_0 = D_1 - b_1B_1, \quad N = D_2 - 3B_2 - c_1B_1
\]
\[
R = M_0 + N_0\rho, \quad S^2 = 1 - 2b_1\rho \cos \gamma + b_1^2\rho^2
\]

the results are

\[
\sin \phi = \left( \sqrt{\rho} R \sin \gamma \right) \left( \cos \gamma - b_1\rho \right) \sqrt{S^2 - \rho R^2} \cdot S^{-2}
\]
\[
\cos \phi = \left( -\sqrt{\rho} R \left( \cos \gamma - b_1\rho \right) \right) \left( \sin \gamma \sqrt{S^2 - \rho R^2} \right) \cdot S^{-2}
\]

and

\[
\left( (M + \rho N) S^2 + \rho R \left( (3 + b_1) \sin \gamma + c_1 \cos \gamma - c_1 b_1 \rho \right) \right) \sqrt{\rho}
\]
\[
= \left\{ -3\rho^2 b_1 + \rho \left( (3 - b_1) \cos \gamma - c_1 \sin \gamma \right) + 1 \right\} \sqrt{S^2 - \rho R^2}.
\]

By squaring the last equation we obtain a polynomial of degree 7 and therefore up to 7 positive roots for $\rho$ are possible. Each root fixes the sign of the two square roots which appear in the formula and therefore $\phi$ is determined uniquely. Therefore we conclude that up to 7 families can bifurcate from the origin. Of course, in order to prove that these families continue to exist for $\epsilon \neq 0$ we have to show that the corresponding Jacobian is nonzero at $\epsilon = 0$. But the Jacobian for the general case is a rather long expression and will not be reproduced here.

The case $k = -3$ with proper notation leads to the same formulas and therefore provides nothing new. Again up to 7 families of long period orbits can bifurcate from the origin.

7. Resonance Cases in the Liapunov Center Theorem

We want to apply the results of the last section to the Liapunov center theorem when the linearized system has two pairs of imaginary eigenvalues with one being an integral multiple of the other. A more detailed analysis has been done
already in [4] and [10] so that the following can be viewed as an example illustrating the previous theory. We will restrict ourselves to an analytic Hamiltonian system with two degrees of freedom. The Hamiltonian function starts with

$$H = \frac{\lambda}{2} (q_1^2 + p_1^2) + \frac{\lambda k}{2} (q_2^2 + p_2^2) + \cdots.$$ 

After a change of time if necessary we can assume that $\lambda = -1$. We again work with complex coordinates $x_j = q_j + ip_j$, $y_j = q_j - ip_j$, $j = 1, 2$ which we combine into the column vector $z = (x_1, x_2, y_1, y_2)^T$. The corresponding Hamiltonian function is

$$K_0(z) = -2iH(z) = i(x_1y_1 + kx_2y_2 + \cdots) = iK(z). \quad (15)$$

Real solutions are only given if $\bar{x}_j = y_j$, $j = 1, 2$ and in that case the function $K(z)$ as defined above is of course real valued. In accordance with section 3 we consider the system

$$\dot{z} = (\mu \mathcal{F} + i \mathcal{J}) \text{grad} K$$

where $\mathcal{J} = (-1, 0)$ is the standard symplectic matrix and $\mathcal{F} = (0, 1)$ destroys the conservative character of the system. Since periodic orbits occur only for $\mu = 0$ the results of section 6 become relevant for Hamiltonian systems.

In the case $|k| \geq 4$ the terms up to order four in the Hamiltonian function (15) have the following normal form

$$K_0(z) = i(x_1y_1 + kx_2y_2 + \frac{1}{2} Rx_1^2y_1^2 + Sx_1y_1x_2y_2 + \frac{1}{2} Tx_2^2y_2^2 + \cdots)$$

with $R$, $S$ and $T$ real (see [11]). The condition (10) for the existence of a family of long periodic orbits reduces to $\mu I = S - kR \neq 0$.

In the case $k = 2$ the Hamiltonian (15) with terms up to order three in normal reads

$$K_0(z) = i(x_1y_1 + 2x_2y_2 + Dx_1^2y_2^2 + Dy_1^2y_2^2 + \cdots)$$

where $D = D_0 e^{i\theta}$ is some complex number. Provided that $D_0 \neq 0$ the determining equation (13) is $x^2 - 4 = 0$ which has the positive solution $x = 2$. At this value the Jacobian (14) is equal to $-4$ and therefore nonzero. This guarantees the existence of two families of long periodic orbits.

In the case $k = -2$ the Hamiltonian (15) starts with the following terms in normal form

$$K_0(z) = i(x_1y_1 - 2x_2y_2 + Dx_1^2x_2 + Dy_1^2y_2 + \cdots).$$

Provided that $D \neq 0$ the determining relation (13) is $x^2 + 4x + 4 = 0$ which has no positive solutions.
This case is similar to the example given in [11] where Siegel shows that when \( D \neq 0 \) then there are no other periodic solutions, except those given by Liapunov's theorem for the eigenvalue \( 2i \). Thus it is apparent that the higher order terms have to satisfy certain conditions if we want to establish the existence of long periodic solutions when the linearized system exhibits resonance. Even if all solutions are \( 2\pi \) periodic in the linearized system then the nonlinear Hamiltonian system does not necessarily have any periodic solutions as the following example demonstrates. We give the Hamiltonian function already in normal form using complex coordinates

\[
K = i(x_1y_1 - x_2y_2 + (x_1x_1 + x_2y_2)(y_1y_1 + y_2y_2)).
\]

It is seen at once that neither \( x_1 = y_1 = 0 \) nor \( x_2 = y_2 = 0 \) contain any periodic solutions. Therefore we can use the polar coordinates \( x_j = y_j = \sqrt{\rho_j e^{i\theta_j}}, j = 1, 2 \) and the transformed Hamiltonian function reads

\[
H = \rho_1 - \rho_2 + 2(\rho_1\rho_2)^{1/2}(\rho_1 + \rho_2) \cos (\theta_1 + \theta_2).
\]

The differential equations for \( \rho_1 \) and \( \rho_2 \) are

\[
\ddot{\rho}_1 = \ddot{\rho}_2 = 2(\rho_1 + \rho_2)(\rho_1^2 + 6\rho_1\rho_2 + \rho_2^2)
\]

and therefore no periodic solutions are possible.

ACKNOWLEDGMENTS

The author would like to express his gratitude to Professor K. R. Meyer whose suggestions simplified some proofs and to Professor J. A. Yorke who made the author aware of the relationship between the theorems of Hopf and Liapunov.

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