

# Central Configurations and Relative Equilibria for the $N$ -Body Problem

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## 1 Introduction

Given are  $N$  point particles of mass  $m_j \in \mathbf{R}^+$  at the positions  $q_j \in \mathbf{R}^3$ , with the velocities  $\dot{q}_j \in \mathbf{R}^3$  for  $j = 1, \dots, N$ . The bodies attract each other in accordance with Newton's law. The problem consists in describing their future motion. With appropriate units for time, distance, and mass, the equations of motion have the form

$$m_j \ddot{q}_j = \frac{\partial U}{\partial q_j}, \quad j = 1, \dots, N. \quad (1)$$

The function

$$U(q) = \sum_{i < j} \frac{m_i m_j}{|q_i - q_j|}$$

will be called the Newtonian potential, although actually it is the negative of the potential. The equations of motion are then

$$\ddot{q}_j = \sum_{i \neq j} \frac{m_i (q_i - q_j)}{|q_i - q_j|^3}.$$

In matrix form the equations are given by

$$\ddot{q} = M^{-1} \frac{\partial U}{\partial q}$$

with  $q = (q_1, \dots, q_N) \in \mathbf{R}^{3N}$  and

$$M = \text{diag}(m_1, m_1, m_1, \dots, m_N, m_N, m_N).$$

The differential equations allow 10 integrals. Six of them come from the fact that the center of mass of the system moves with constant velocity along a straight line. One can account for these integrals by using a coordinate system where the center of mass resides at the origin at all times, that is,  $m_1 q_1 + \dots + m_N q_N = 0$ . Three integrals are due to the preservation of the angular momentum for the entire system, that is,  $\sum m_j q_j \times \dot{q}_j = \text{const}$ , and an additional integral corresponds to the preservation of the total energy, that is,

$$\frac{1}{2} \sum_{i=1}^N m_i |\dot{q}_i|^2 + U(q) = \text{const}.$$

It is possible to reduce the order of the system of differential equations by utilizing these last 4 integrals, but the resulting set of equations is too complicated, so this reduction is seldom done.

Relative equilibria and central configurations seem to be the only solutions for the general  $N$ -body problem that can be given in closed form. In both cases one assumes that at all times the acceleration vector for each body points toward the origin. For central configurations one assumes that the motion takes place on straight lines going through the origin. In other word, one looks for solutions to (1) of the form

$$q_j = r(t)Q_j$$

with  $r(t)$  a scalar function and  $Q_j$  constant vectors. Such a solution can be found if  $r(t)$  satisfies Kepler's equation,

$$\ddot{r} + \lambda/r^2 = 0, \quad (2)$$

for some constant  $\lambda$  and if the vectors  $Q_j$ ,  $j = 1, \dots, N$ , are a solution to the algebraic set of equations

$$\sum_{i < j} \frac{Q_i - Q_j}{|Q_i - Q_j|^3} = \lambda Q_j, \quad j = 1, \dots, N. \quad (3)$$

In matrix form these equations are

$$M^{-1} \frac{\partial U}{\partial Q} = \lambda Q,$$

and since the potential function  $U(q)$  is a homogeneous function of degree  $-1$ , it follows from Euler's theorem that

$$\lambda = -\frac{U(q)}{q^t M q}.$$

The constant  $\lambda$  can also be viewed as a Lagrange multiplier so that the solutions of (3) can be viewed as the extrema of the potential  $U(q)$  under the condition that the moment of inertia,

$$I(q) = \frac{1}{2} \sum_{i=1}^N m_i |q_i|^2,$$

remains constant. This point of view will be explored later on. It will allow us to use bifurcation theory to find new central configurations near very special solutions of (3). Only for  $N \leq 3$  can (3) be analyzed completely.

One type of solution for (2) is

$$r(t) = (-9\lambda/2)^{1/3} t^{2/3},$$

indicating that all bodies will collide at time  $t = 0$  at the origin. Although this solution is very special, it is interesting, because it has been shown that any other collision orbits approach it asymptotically.

## 2 Dziobek's Coordinates

Our approach uses bifurcation theory to search for new relative equilibria or central configurations near known ones. In order to do this, we have to find degenerate central configurations. In this context a central configuration is called degenerate if it is not isolated. But in fact, no configuration is isolated, because near each configuration is another one obtained from the first by a simple rotation. This is a manifestation of the fact that we have not yet accounted for the integrals which give the preservation of the angular momentum. For this reason one introduces equivalence classes of central configurations. Two configurations are called equivalent if they can be transformed into each other by a rotation around the center of mass. Degeneracy then means degeneracy among equivalence classes, that is,  $\text{mod}SO(2)$  or  $\text{mod}SO(3)$ .

Difficulties of this nature can often be avoided with the choice of an appropriate coordinate system. The mutual distances between the different bodies is such a choice. The distances determine a configuration, but they do not change with a rotation. They have been used by Dziobek [1] already and were exploited further in [3] and [6]. Let  $r_{ij}$  be the distance between the  $i$ th and the  $j$ th body. The potential function is then

$$U = \sum_{i < j} \frac{m_i m_j}{r_{ij}},$$

and with the center of mass at the origin, the moment of inertia is

$$I = \frac{1}{2M} \sum_{i < j} m_i m_j r_{ij}^2,$$

with  $M = \sum m_i$  the total mass of the system.

In the following sections we will use the coordinates of Dziobek to derive some of the classical results for the three-, four-, and five-body problems, and we include our results, which were obtained with the help of bifurcation analysis.

## 3 Configurations for the Three-Body Problem

For three bodies, the distances  $r_{12}$ ,  $r_{23}$ , and  $r_{13}$  determine the configuration up to a reflection, provided that the obvious geometric constraint in the form of the triangle inequality is met.

It was mentioned in section 1 that central configurations are extrema of  $U$  under the condition that the moment of inertia is constant. It means that they are the extrema of the function  $U + \lambda(I - I_0)$  or the solution to the equations

$$-\frac{m_i m_j}{r_{ij}^2} + \frac{\lambda}{M} m_i m_j r_{ij} = 0, \quad 1 \leq i < j \leq 3,$$

$$\sum_{1 \leq i < j \leq 3} m_i m_j r_{ij}^2 - 1 = 0,$$

where we have selected  $I_0 = 1/2M$ . The solutions to these equations give at once the equilateral solutions of Lagrange:

$$r_{12} = r_{23} = r_{13} = (m_1 m_2 + m_2 m_3 + m_3 m_1)^{-1/2}.$$

On the other hand, the collinear solutions of Euler are not even indicated due to the fact that the given potential function does not have an extrema at the collinear solutions. The collinear solutions can be found if we search extrema under the constraint that all three bodies lie on a line.

Although the collinear solutions can be found directly from the differential equations, we present their derivation with the help of the coordinates of Dziobek for completeness sake and to illustrate the method we will use for the problem with four and five bodies. The three bodies will lie on a straight line if we insist that the area of the triangle they form is zero. The determinant

$$F(r_{12}^2, r_{13}^2, r_{23}^2) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 \end{vmatrix}$$

is related to the area  $A$  of the triangle via  $F = 32A^2$ . When  $F = 0$  it has the property that

$$\frac{\partial F}{\partial r_{ij}^2} = 4\delta_i \delta_j,$$

where  $\delta_i$  and  $\delta_j$  are the lengths of the sides opposite to the points  $P_i$  and  $P_j$ , respectively. These lengths have to be taken with the appropriate orientation, that is, relative to an outward normal of the triangle. Selecting that  $P_2$  lies between  $P_1$  and  $P_3$ , we require that

$$r_{13} = r_{12} + r_{23},$$

from which it follows that

$$\frac{\partial F}{\partial r_{12}^2} = -4r_{13}r_{23}, \quad \frac{\partial F}{\partial r_{23}^2} = -4r_{12}r_{13}, \quad \frac{\partial F}{\partial r_{13}^2} = 4r_{12}r_{23}.$$

In order to find the extrema of  $U$  under the additional constraint  $F = 0$ , let  $\sigma$  be another Lagrangian multiplier, so that we have to find the extrema of

$$\sum_{1 \leq i < j \leq 3} m_i m_j / r_{ij} + \lambda(I - I_0) + \sigma F.$$

The condition for collinear extrema is then

$$m_i m_j (\lambda r_{ij} - r_{ij}^{-2}) + 2\sigma \frac{\partial F}{\partial r_{ij}^2} = 0, \quad 1 \leq i < j \leq 3,$$

$$I - I_0 = 0, \quad F = 0.$$

The first three equations, slightly modified, are

$$\begin{aligned} m_1 m_2 m_3 (\lambda - r_{12}^{-3}) - 8\sigma m_3 r_{13} r_{23} &= 0, \\ m_1 m_2 m_3 (\lambda - r_{23}^{-3}) - 8\sigma m_1 r_{12} r_{13} &= 0, \\ m_1 m_2 m_3 (\lambda - r_{13}^{-3}) + 8\sigma m_2 r_{12} r_{23} &= 0. \end{aligned}$$

First eliminate  $\sigma$  to obtain

$$\frac{\lambda - r_{12}^{-3}}{m_3 r_{13} r_{23}} = \frac{\lambda - r_{23}^{-3}}{m_1 r_{12} r_{13}} = \frac{-\lambda + r_{13}^{-3}}{m_2 r_{12} r_{23}}$$

and then solve for  $\lambda$  by using the first and the third expressions and then again the second and the third, so that

$$\lambda = \frac{m_2 r_{12}^{-2} + m_3 r_{13}^{-2}}{m_2 r_{12} + m_3 r_{13}} = \frac{m_2 r_{23}^{-2} + m_1 r_{13}^{-2}}{m_2 r_{23} + m_1 r_{13}}.$$

Let  $\rho$  be a factor between 0 and 1 so that

$$r_{12} = \rho r_{13} \quad \text{and} \quad r_{23} = (1 - \rho) r_{13}.$$

Then

$$\lambda r_{13}^3 = \frac{m_2 \rho^{-2} + m_3}{m_2 \rho + m_3} = \frac{m_2 (1 - \rho)^{-2} + m_1}{m_2 (1 - \rho) + m_1}$$

and in the interval  $0 < \rho < 1$ , the difference

$$f(\rho) = \frac{m_2 \rho^{-2} + m_3}{m_2 \rho + m_3} - \frac{m_2 (1 - \rho)^{-2} + m_1}{m_2 (1 - \rho) + m_1}$$

is a monotonically decreasing function. It tends to  $+\infty$  for  $\rho \rightarrow 0$  and to  $-\infty$  for  $\rho \rightarrow 1$ . Therefore, there exists a unique solution for  $\rho$  in  $0 < \rho < 1$ . The solution is the only real solution of the following fifth-order polynomial

$$\begin{aligned} (m_1 + m_3)\rho^5 - (2m_1 + 3m_3)\rho^4 + (m_1 + 2m_2 + 3m_3)\rho^3 \\ - (m_1 + 3m_2)\rho^2 + (2m_1 + 3m_2)\rho - m_1 - m_2 = 0. \end{aligned}$$

The value of  $r_{13}$  is found from the condition that the moment of inertia is preserved. The other collinear solutions are found by interchanging the points  $P_1$ ,  $P_2$ , and  $P_3$ .

## 4 Configurations in the Four-Body Problem

Between four bodies there are six mutual distances. Assume that these distances are given and that they meet the necessary geometric constraint so that they define a tetrahedron in  $\mathbf{R}^3$ . We are looking for extrema of

$$\sum_{1 \leq i < j \leq 4} m_i m_j \left( 1/r_{ij} + \frac{\lambda}{2}(r_{ij}^2 - 1) \right).$$

They follow from the six equations

$$m_i m_j (\lambda - r_{ij}^{-3}) = 0, \quad 1 \leq i < j \leq 4,$$

and from the constraint

$$\sum m_i m_j (r_{ij}^2 - 1) = 0.$$

It is at once apparent that the only solution to these equations determines a regular tetrahedron in  $\mathbf{R}^3$ . If we are looking for planar solutions we have to impose the additional constraint that the volume of the tetrahedron is zero, that is, that

$$F = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 0. \quad (4)$$

Similar to the collinear for the three-body problem we have to look for the extrema of

$$U + \lambda(I - I_0) + \frac{\sigma}{32}F.$$

It leads to the six equations

$$m_i m_j (\lambda - r_{ij}^{-3}) + \sigma A_i A_j = 0 \quad (5)$$

together with

$$I - I_0 = 0 \quad (6)$$

and

$$F = 0. \quad (7)$$

As explained in Appendix A,  $A_i$  is the oriented area of the triangle, which does not include the point  $P_i$ . When the masses are given, the above eight equations can be used to determine the eight quantities  $\lambda$ ,  $\sigma$ ,  $r_{12}, \dots, r_{34}$ . Unfortunately these equations are nonlinear, so little is known about the solution manifolds in the general case. Instead special cases are considered or questions of a different nature are asked. One of these questions is, Given a configuration in the plane, is it possible to find four positive masses with which this configuration can be realized?

To answer the above question, we display the first six equations in the following form:

$$\begin{aligned} m_1 m_2 (r_{12}^{-3} - \lambda) &= \sigma A_1 A_2, & m_3 m_4 (r_{34}^{-3} - \lambda) &= \sigma A_3 A_4, \\ m_1 m_3 (r_{13}^{-3} - \lambda) &= \sigma A_1 A_3, & m_2 m_4 (r_{24}^{-3} - \lambda) &= \sigma A_2 A_4, \\ m_1 m_4 (r_{14}^{-3} - \lambda) &= \sigma A_1 A_4, & m_2 m_3 (r_{23}^{-3} - \lambda) &= \sigma A_2 A_3. \end{aligned}$$

The equations have been grouped so that when they are multiplied together pairwise, their right-hand sides are identical. This leads to the well-known condition

$$(r_{12}^{-3} - \lambda)(r_{34}^{-3} - \lambda) = (r_{13}^{-3} - \lambda)(r_{24}^{-3} - \lambda) = (r_{14}^{-3} - \lambda)(r_{23}^{-3} - \lambda). \quad (8)$$

This is a necessary and sufficient condition for solving equations (5)–(7) for the four masses, when the distances are given. It is sufficient because (8) shows that one can solve for  $\lambda$ , from which the ratio of the masses can be found. The condition (8) is necessary, as it guarantees that the three expressions for  $\lambda$  will be equal. They follow from (8) by solving different pairs of equations, and they are with the abbreviation  $s_{ij} = r_{ij}^{-3}$ :

$$\lambda = \frac{s_{12}s_{34} - s_{13}s_{24}}{s_{12} + s_{34} - s_{13} - s_{24}} = \frac{s_{13}s_{24} - s_{14}s_{23}}{s_{13} + s_{24} - s_{14} - s_{23}} = \frac{s_{14}s_{23} - s_{12}s_{34}}{s_{14} + s_{23} - s_{12} - s_{34}}.$$

From the different ratios of two masses that can be found from (5), we obtain the following list:

$$\frac{m_1 A_2}{m_2 A_1} = \frac{s_{23} - \lambda}{s_{13} - \lambda} = \frac{s_{24} - \lambda}{s_{14} - \lambda} = \frac{s_{23} - s_{24}}{s_{13} - s_{14}}, \quad (9)$$

$$\frac{m_1 A_3}{m_3 A_1} = \frac{s_{23} - \lambda}{s_{12} - \lambda} = \frac{s_{34} - \lambda}{s_{14} - \lambda} = \frac{s_{23} - s_{34}}{s_{12} - s_{14}}, \quad (10)$$

$$\frac{m_1 A_4}{m_4 A_1} = \frac{s_{24} - \lambda}{s_{12} - \lambda} = \frac{s_{34} - \lambda}{s_{13} - \lambda} = \frac{s_{24} - s_{34}}{s_{12} - s_{13}}, \quad (11)$$

$$\frac{m_2 A_3}{m_3 A_2} = \frac{s_{13} - \lambda}{s_{12} - \lambda} = \frac{s_{34} - \lambda}{s_{24} - \lambda} = \frac{s_{13} - s_{34}}{s_{12} - s_{24}}, \quad (12)$$

$$\frac{m_2 A_4}{m_4 A_2} = \frac{s_{14} - \lambda}{s_{12} - \lambda} = \frac{s_{34} - \lambda}{s_{23} - \lambda} = \frac{s_{14} - s_{34}}{s_{12} - s_{23}}, \quad (13)$$

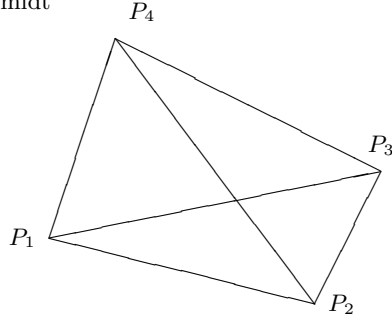
$$\frac{m_3 A_4}{m_4 A_3} = \frac{s_{14} - \lambda}{s_{13} - \lambda} = \frac{s_{24} - \lambda}{s_{23} - \lambda} = \frac{s_{14} - s_{24}}{s_{13} - s_{23}}. \quad (14)$$

Condition (8) guarantees only that the mass ratios can be computed. It does not say that the masses will be positive. With the help of equations (9)–(14) this additional information can be obtained.

Two cases have to be considered separately. These are the convex and the concave configurations. In the concave case one body lies inside the triangle formed by the other three bodies. In the convex case this does not occur. The degenerate case where three bodies lie on a line leads to one mass being zero as long as not all four bodies lie on a line. This follows from the following argument. Assume that the three bodies with nonzero masses  $m_1$ ,  $m_2$ , and  $m_3$  lie on a line. From the assumptions it follows that for the triangle that does not contain the fourth body we must have  $A_4 = 0$ . Since none of the other triangles are degenerate, it follows from (11), (13), and also (14) that  $s_{14} = s_{24} = s_{34}$ . This would require that at least two bodies coincide. If this is not the case, then at least one of the four masses has to be zero.

We start with the convex case and assume that the bodies are arranged as in Figure 1. The signs of the four areas are then

$$\begin{aligned} A_1 &= \Delta P_2 P_3 P_4 > 0, \\ A_2 &= \Delta P_1 P_4 P_3 < 0, \end{aligned}$$

**Fig. 1.** Convex configuration.

$$\begin{aligned} A_3 &= \Delta P_1 P_2 P_4 > 0, \\ A_4 &= \Delta P_1 P_3 P_2 < 0. \end{aligned}$$

If we assume that  $s_{12} - \lambda > 0$  and ask for positive masses, we find from equations (9)–(14) that

$$s_{12}, s_{23}, s_{34}, s_{14} > \lambda > s_{13}, s_{24}.$$

With the notation  $\lambda = \rho^{-3}$ , the above inequality can be written in terms of distances

$$r_{12}, r_{23}, r_{34}, r_{14} < \rho < r_{13}, r_{24}. \quad (15)$$

It means that all exterior sides have to be smaller than the diagonals. If we had assumed that  $s_{12} - \lambda < 0$ , then the inequalities in (15) would be reversed and the configuration could not be realized geometrically. If we make the assumption that  $r_{12}$  is the smallest side, then the last expressions in (10) and in (13) give the conditions that  $s_{12} > s_{23} > s_{34}$  and  $s_{12} > s_{14} > s_{34}$ , or equivalently that  $r_{12} < r_{23} < r_{34}$  and  $r_{12} < r_{14} < r_{34}$ . It means that the longest and the shortest sides have to lie opposite each other. This result can already be found in [1].

Limits on the ratios of the diagonals of a feasible central configuration can be obtained by starting the construction of the quadrilateral with the diagonal of length  $r_{13} > \rho$ . Draw circles of radius  $\rho$  around  $P_1$  and  $P_3$ , respectively. The other two masses have to lie in the common area of intersection. Since also  $r_{24} > \rho$ , these masses have to be outside the two circles around  $Q_1$  and  $Q_2$ . Since

$$\left(\frac{r_{13}}{2}\right)^2 + \left(\frac{\overline{Q_1 Q_2}}{2}\right)^2 = \rho^2$$

but  $\rho < r_{24} < \overline{Q_1 Q_2}$ , we find that

$$r_{13} < \sqrt{3}\rho < \sqrt{3}r_{24}.$$

By an analogous argument

$$r_{24} < \sqrt{3}\rho < \sqrt{3}r_{13}.$$

Together, then, we have the following theorem, also found in [2].

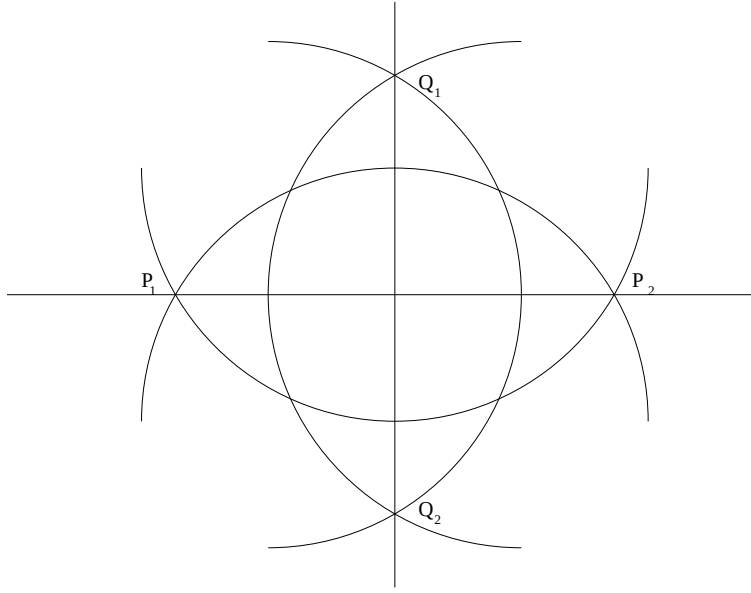


**Fig. 2.** Limits on the ratio of the diagonals.

**Theorem 1.** *For a convex central configuration with positive masses the ratio of the diagonals is limited by*

$$1/\sqrt{3} < r_{13}/r_{24} < \sqrt{3}.$$

*Furthermore, the size of the interior angles is restricted to the interval from 30 to 120 degrees.*

The last statement of the theorem can be verified easily by looking at the extreme cases which with Figure 2 could be drawn and still meeting the condition (15).

For concave configurations, let  $m_4$  be the interior mass. With  $A_4 < 0$  and  $A_1, A_2, A_3 > 0$  we find from (9)–(14) that

$$s_{12}, s_{23}, s_{13} < \lambda < s_{14}, s_{24}, s_{34}$$

or

$$r_{12}, r_{23}, r_{13} > \rho > r_{14}, r_{24}, r_{34}.$$

It means that all interior edges are longer than the exterior edges. If we assume the ordering  $r_{12} > r_{23} > r_{13}$  for the lengths of the exterior edges, then we find that  $r_{34} > r_{14} > r_{24}$  and we can state the following theorem.

**Theorem 2.** *In a concave configuration with four positive masses, the longest exterior side lies opposite the longest interior edge. The same statement also holds for the shortest exterior and interior edges.*

By considering each exterior edge one at a time, we can even state in which of the half-planes defined by the perpendicular bisector the body  $m_4$  has to lie. If its mass is to be positive, it has to lie in the half-plane, which does not include the third exterior vertex. The interior body is therefore restricted to the intersection of three half-planes. Since the three perpendicular bisectors of a triangle meet at one point, this intersection is not empty, but it may lie outside of the given triangle. This will not happen if the center of the circumscribing circle lies inside the given triangle. This restriction can be expressed algebraically by the inequalities  $a^2 < b^2 + c^2$ , where  $a$ ,  $b$ , and  $c$  are any permutation of  $r_{12}$ ,  $r_{23}$ , and  $r_{13}$ .

Equations (9)–(14) allow one also to look for degenerate configurations. Degeneracy is indicated by indeterminate expressions on the right-hand sides. Assuming that  $m_4$  cannot be determined, we obtain at once that  $s_{12} = s_{23} = s_{13}$  and  $s_{14} = s_{24} = s_{34}$ . It represents an equilateral triangle with three equal masses  $m_1 = m_2 = m_3$  at the vertices and a variable mass  $m_4$  at its center. Let the exterior masses be of unit value and  $r_{12} = r_{23} = r_{13} = \sqrt{3}$ , so that  $r_{14} = r_{24} = r_{34} = 1$ . We will vary the interior mass  $m_4$  and look for a value  $m_c$  so that the configuration becomes degenerate and bifurcation analysis can be applied.

In order to explain the details of our calculations, we introduce the eight-dimensional vector

$$z = (\lambda, \sigma, r_{12}, r_{13}, r_{23}, r_{14}, r_{24}, r_{34}).$$

Set  $m_1 = m_2 = m_3 = 1$  and let  $m_4 = m$  be indeterminate so far. The eight equations of (5), (6), and (7) are

$$\begin{aligned} F &= 0, \\ I - I_0 &= 0, \\ (\lambda - r_{12}^{-3}) + \sigma A_1 A_2 &= 0, \\ (\lambda - r_{13}^{-3}) + \sigma A_1 A_3 &= 0, \\ (\lambda - r_{23}^{-3}) + \sigma A_2 A_3 &= 0, \\ m(\lambda - r_{14}^{-3}) + \sigma A_1 A_4 &= 0, \\ m(\lambda - r_{24}^{-3}) + \sigma A_2 A_4 &= 0, \\ m(\lambda - r_{34}^{-3}) + \sigma A_3 A_4 &= 0. \end{aligned} \tag{16}$$

Denote these equations by  $W(z, m) = 0$ . For  $I_0 = (9 + 3m)/2$ , the equilateral triangular family is given by  $z = a$ , where

$$a = \left( \frac{3m + \sqrt{3}}{6(3m + 9)}, \frac{4(9 - \sqrt{3})m}{27(3 + m)}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 1, 1, 1 \right).$$

Thus  $W(a, m) = 0$  for all  $m$ . With the aid of a computer algebra program we find that the Jacobian

$$\left| \frac{\partial W}{\partial z} \right|_{z=a} = (133 - 60\sqrt{3})m_c^2(249m_c - 64\sqrt{3} - 81)^2/20667$$

is positive definite except for

$$m_4 = m_c = \frac{81 + 64\sqrt{3}}{249},$$

a value already found by Palmore [5]. For  $m_4 = m_c$  the rank of the Hessian reduces by 2. A nonzero subdeterminant of the Hessian is obtained by deleting the last two rows and columns. By the implicit function theorem, we can thus solve the first six equations for  $\lambda, \sigma, r_{12}, r_{13}, r_{23}, r_{14}$  in terms of  $r_{24}, r_{34}$  for  $m$  near  $m_c$ .

Set  $m = m_c + \varepsilon$  and  $z = a + \varepsilon b + \varepsilon^2 c + \dots$  and solve the set of equations  $W(z, m) = 0$  order by order in  $\varepsilon$ . Let  $\beta = (b_1, b_2, b_3, b_4, b_5, b_6)$ ; then, from first-order terms, we get  $\beta = \beta(b_7, b_8)$ , with the equations  $W_7, W_8$  at order 1 satisfied automatically. Similarly, for second-order terms let  $\gamma = (c_1, \dots, c_6)$ , and they can be found as functions of  $b_7, b_8, c_7$ , and  $c_8$ . The last two equations now deliver the bifurcation equations. They are in factored form:

$$\begin{aligned} (2b_7 + b_8)(b_8 + p) &= 0, \\ (b_7 + 2b_8)(b_7 + p) &= 0, \end{aligned} \tag{17}$$

with

$$p = \frac{4531167 - 3089347\sqrt{3}}{18889832}.$$

There are three sets of nontrivial solutions:

$$\begin{aligned} b_7 &= -p, \quad b_8 = -p, \\ b_7 &= -p, \quad b_8 = 2p, \\ b_7 &= 2p, \quad b_8 = -p. \end{aligned}$$

They are a consequence of the threefold symmetry of the problem, and we can concentrate on the first solution. The Jacobian of (17) at that solution is  $-p^2 \neq 0$  and higher order terms can be found. It means that the last two distances  $r_{24}$  and  $r_{34}$  can be expanded as series in  $\varepsilon$  and are  $r_{24} = r_{34} = 1 - \varepsilon p + \mathcal{O}(\varepsilon^2)$ . Previously we had found the first six variables, so that  $z = a + \varepsilon b + \mathcal{O}(\varepsilon^2)$  could be computed order by order.

For the distances within the triangle we have through order 1

$$\begin{aligned} r_{12} = r_{13} &= \sqrt{3} - \frac{81 + 64\sqrt{3}}{83}p\varepsilon + \dots, \\ r_{23} &= \sqrt{3} + \frac{81 + 64\sqrt{3}}{83}p\varepsilon + \dots, \\ r_{14} &= 1 + 2p\varepsilon + \dots, \\ r_{24} = r_{34} &= 1 - p\varepsilon + \dots. \end{aligned}$$

## 5 Configurations in the Five-Body Problem

With five bodies there are ten mutual distances  $r_{ij}$ ,  $1 \leq i < j \leq 5$ . Three constraints are needed if the distances define a planar configuration. Only one constraint is needed if the configuration is to be realized in  $\mathbf{R}^3$ . The two cases will be discussed separately, and we will start with the planar case.

Let  $F(i, j, k, l)$  be the determinant given in (4), with  $(i, j, k, l)$  replacing  $(1, 2, 3, 4)$  in that order. We choose the following constraints to insure that the configuration is planar:

$$F_1 = F(1, 2, 3, 4), \quad F_2 = F(1, 2, 3, 5), \quad F_3 = F(2, 3, 4, 5).$$

Consider

$$V = U + \lambda(I - I_0) + \sigma_1 F_1 + \sigma_2 F_2 + \sigma_3 F_3$$

so that  $V$  is a function of the 14 variables given in the vector

$$z = (\lambda, \sigma_1, \sigma_2, \sigma_3, r_{12}, r_{13}, r_{23}, r_{14}, r_{2,4}, r_{34}, r_{15}, r_{25}, r_{35}, r_{45}).$$

The equations to be solved are  $\partial V / \partial z = 0$ . They are similar to those in (16) and again denoted by  $W(z, m) = 0$ . With four equal masses  $m_1 = m_2 = m_3 = m_4 = 1$  at the vertices of a square, we have a solution to these equations with

$$z = a = \left( \frac{1 + 2\sqrt{2} + 4m}{4m + 16}, \frac{-1 + 2\sqrt{2} + m(2\sqrt{2} - 4)}{256 + 64m}, \frac{(15 - 2\sqrt{2})m}{128 + 32m}, \right. \\ \left. \frac{(15 - 2\sqrt{2})m}{128 + 32m}, \sqrt{2}, 2, \sqrt{2}, \sqrt{2}, 2, \sqrt{2}, 1, 1, 1, 1 \right)$$

with any arbitrary mass  $m_5 = m$  at the center of the square. The Jacobian of the system of equations is then

$$73728(67\sqrt{2} - 84)(4m + 1)(12m - 13 - 11\sqrt{2})^2,$$

which is nonzero for all positive masses except for one critical value

$$m = m_c = \frac{13 + 11\sqrt{2}}{12}.$$

A nonzero subdeterminant can be found by deleting the last two rows and columns. By the implicit function theorem we can thus solve the first 12 equations of  $W(z, m) = 0$  for  $m$  near  $m_c$  in terms of  $r_{35}$  and  $r_{45}$ . Set

$$m = m_c + \delta \varepsilon^2 \quad \text{with} \quad \delta = \pm 1 \quad \text{and let} \quad z = a + \varepsilon b + \varepsilon^2 c + \varepsilon^3 d + \dots$$

Set  $Z = (W_{13}, W_{14})$  to be the last two equations of  $W(z, m)$  and  $u = (r_{35}, r_{45})$  so that the problem is reduced to the study of the bifurcation equations

$$Z(u, \varepsilon) = 0.$$

The first nonzero terms occur at order  $\varepsilon^3$ . These are, except for some constants,

$$(\delta q_1 - q_2 b_{35}^2 + q_3 b_{45}^2) b_{35} = 0, \quad (18)$$

$$(\delta q_1 + q_3 b_{35}^2 - q_2 b_{45}^2) b_{45} = 0, \quad (19)$$

where

$$q_1 = 122691244 + 411830784\sqrt{2},$$

$$q_2 = 291093160040 + 211659410269\sqrt{2},$$

$$q_3 = 1204699032744 + 860002887155\sqrt{2}.$$

For  $\delta = +1$ , the solutions are

$$b_{35} = 0, \quad b_{45} = \pm\sqrt{q_1/q_2}, \quad \text{or} \quad b_{35} = \pm\sqrt{q_1/q_2}, \quad b_{45} = 0.$$

For  $\delta = -1$ , the solutions are

$$b_{35} = b_{45} = \pm\sqrt{q_3 - q_2} \quad \text{and} \quad b_{35} = -b_{45} = \pm\sqrt{q_3 - q_2}.$$

At first order in  $\varepsilon$ , we had found

$$b_{12} = -b_{34} = \frac{82 + 75\sqrt{2}}{62}(b_{35} + b_{45}),$$

$$b_{14} = -b_{23} = \frac{82 + 75\sqrt{2}}{62}(b_{35} - b_{45}),$$

$$b_{13} = b_{24} = 0, \quad b_{15} = -b_{35}, \quad b_{25} = -b_{45},$$

so that the solution for  $\delta = +1$  gives central configurations that look like a kite, and  $\delta = -1$  corresponds to central solutions in the form of a trapezoid; see Figure 3. The multiple choice in the signs accounts for the symmetries of the square configuration. A configuration in the shape of a rhombus would only occur for a negative mass,  $m_5 = -0.25$ .

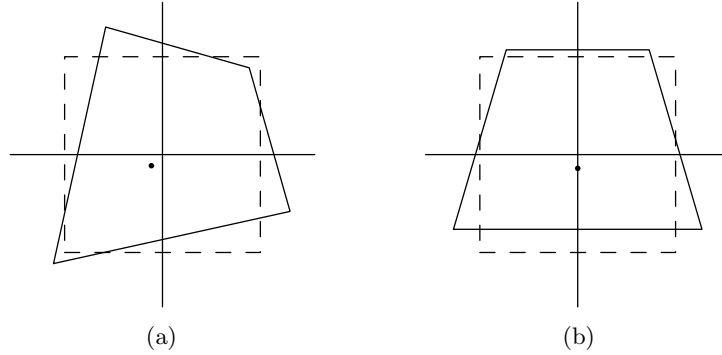


**Fig. 3.** Bifurcations for the five-body problem: (a)  $\delta = -1$ ; (b)  $\delta = +1$ .

## 6 Palmore's Coordinates

The number of constraints which insures that a configuration can be realized geometrically in  $\mathbf{R}^2$  or  $\mathbf{R}^3$  increases quadratically with the number of the bodies. Therefore, mutual distances as coordinates become less attractive. Instead, Cartesian coordinates are used. The complexity of the problem is reduced by starting with regular configurations and looking for configurations that might bifurcate from them.

Place  $N$  bodies of unit mass at the vertices of a regular  $N$ -gon and a mass of size  $m$  at the origin. We will vary  $m$  to search for a new configuration nearby. Let  $w = e^{2\pi i/N}$ ; then  $q_j = w^j$ ,  $j = 0, \dots, N-1$ , are the positions of the  $N$  unit masses and  $q_N = 0$  is the position of the body at the origin in this  $(N+1)$ -body problem. Configurations nearby are described most conveniently by a coordinate system used by Palmore [5] and possibly by others before him. Let  $\Omega = (\omega^0, \omega^1, \dots, \omega^{N-1}, 0)^T$  be the position vector for the  $N+1$  bodies in the regular polygon configuration and  $q = \Omega + Az$  be the position vector for the configuration in the vicinity. Here  $z = (z_0, \dots, z_N)^T$  and

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \dots & \omega^{N-1} & 1 \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \omega^{(N-1)} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} & 1 \\ 0 & 0 & 0 & \dots & 0 & -N/m \end{pmatrix}.$$

The individual positions of the bodies are given in complex form by

$$q_j = \omega^j + \sum_{i=0}^{N-1} \omega^{ij} z_i + z_N, \quad j = 0, \dots, N-1,$$

and for the body in the center by

$$q_N = -\frac{N}{m} z_N.$$

The center of mass is found to be

$$\sum_{j=0}^N m_j q_j = \sum_{j=0}^{N-1} \left( \omega^j + \sum_{i=0}^{N-1} (\omega^{ij} z_i + z_N) \right) - m \frac{N}{m} z_N = N z_0.$$

Setting  $z_0 = 0$  fixes the center of mass at the origin.

The first step is to compute the Hessian of the potential function. The calculations are somewhat tedious and error prone, and for this reason we give some of the details here. We write the potential function as

$$U = \sum_{0 \leq i < j \leq N} \frac{m_i m_j}{|q_i - q_j|}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \frac{1}{|q_i - q_{i+j}|} + \sum_{j=0}^{N-1} \frac{m}{|q_j - q_N|} \\
&= U_1 + mU_2.
\end{aligned}$$

With the abbreviation  $d_j = 1 - \omega^j$  and

$$|d_j| = |\omega^{j/2} - \omega^{-j/2}| = 2 \sin j\pi/N,$$

the distances between two bodies of unit mass is

$$\begin{aligned}
|q_i - q_{i+j}| &= \left| \omega^i - \omega^{i+j} + \sum_{k=0}^{N-1} (\omega^{ik} - \omega^{(i+j)k}) z_k \right| \\
&= \left[ d_j d_{-j} + \sum_{k=0}^{N-1} (\omega^{i(k-1)} d_{jk} d_{-j} z_k + \omega^{i(1-k)} d_{-jk} d_j \bar{z}_k) \right. \\
&\quad \left. + \sum_{k,l} \omega^{i(k-l)} d_{jk} d_{-jl} z_k \bar{z}_l \right]^{\frac{1}{2}}.
\end{aligned}$$

When  $z = 0$ , we thus have  $|q_i - q_{i+j}| = |d_j|$  for  $i, j = 0, \dots, N-1$ . We found it easiest to set  $z_j = x_j + \sqrt{-1}y_j$  and to calculate first the derivatives of  $U_1$  with respect to  $x_j$  and  $y_j$ ,  $j = 1, \dots, N-1$ . In this way the results can also be used for the  $N$ -body problem, when the body in the center does not exist.

For  $1 \leq r \leq N-1$ , we find

$$\begin{aligned}
\frac{\partial U_1}{\partial x_r} &= -\frac{1}{4} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \left\{ \omega^{i(r-1)} d_{jr} d_{-j} + \omega^{i(1-r)} d_{-jr} d_j \right. \\
&\quad \left. + \sum (\omega^{i(k-r)} d_{jk} d_{-jr} z_k + \omega^{i(r-k)} d_{jr} d_{-jk} \bar{z}_k) \right\} / |q_i - q_{i+j}|^3,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial U_1}{\partial y_r} &= -\frac{\sqrt{-1}}{4} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \left\{ \omega^{i(r-1)} d_{jr} d_{-j} - \omega^{i(1-r)} d_{-jr} d_j \right. \\
&\quad \left. + \sum (\omega^{i(r-k)} d_{jr} d_{-jk} \bar{z}_k - \omega^{i(k-r)} d_{jk} d_{-jr} z_k) \right\} / |q_i - q_{i+j}|^3.
\end{aligned}$$

We then compute for  $1 \leq s \leq N-1$ , and setting  $z = 0$ ,

$$\begin{aligned}
\frac{\partial^2 U_1}{\partial x_r \partial x_s} &= \frac{3}{8} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \frac{\omega^{i(r+s-2)} d_{jr} d_{js} d_{-j}^2 + \omega^{-i(r+s-2)} d_{-jr} d_{-js} d_j^2}{|d_j|^5} \\
&\quad + \frac{1}{8} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \frac{\omega^{i(s-r)} d_{js} d_{-jr} + \omega^{i(r-s)} d_{jr} d_{-js}}{|d_j|^3},
\end{aligned}$$

$$\begin{aligned} \frac{\partial^2 U_1}{\partial x_r \partial y_s} &= \frac{3\sqrt{-1}}{8} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \frac{\omega^{i(r+s-2)} d_{jr} d_{js} d_{-j}^2 - \omega^{i(2-r-s)} d_{-jr} d_{-js} d_j^2}{|d_j|^5} \\ &\quad + \frac{\sqrt{-1}}{8} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \frac{\omega^{i(s-r)} d_{js} d_{-jr} - \omega^{i(r-s)} d_{jr} d_{-js}}{|d_j|^3}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 U_1}{\partial y_r \partial y_s} &= -\frac{3}{8} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \frac{\omega^{i(r+s-2)} d_{jr} d_{js} d_{-j}^2 + \omega^{-i(r+s-2)} d_{-jr} d_{-js} d_j^2}{|d_j|^5} \\ &\quad + \frac{1}{8} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \frac{\omega^{i(s-r)} d_{js} d_{-jr} + \omega^{i(r-s)} d_{jr} d_{-js}}{|d_j|^3}. \end{aligned}$$

It turns out that most second-order derivatives are zero. Only for a few indices do we find nonzero values. They are, for  $s = r$ ,

$$\frac{\partial^2 U_1}{\partial x_r^2} = \frac{N}{4} \sum_{j=1}^{N-1} \frac{|d_{jr}|^2}{|d_j|^3} = \frac{N}{8} \sum_{j=1}^{N-1} \frac{\sin^2 \pi jr/N}{\sin^3 \pi j/N},$$

$$\frac{\partial^2 U_1}{\partial y_r^2} = \frac{N}{4} \sum_{j=1}^{N-1} \frac{|d_{jr}|^2}{|d_j|^3} = \frac{N}{8} \sum_{j=1}^{N-1} \frac{\sin^2 \pi jr/N}{\sin^3 \pi j/N},$$

and for  $s = N + 2 - r$ ,

$$\begin{aligned} \frac{\partial^2 U_1}{\partial x_r \partial x_s} &= \frac{3N}{8} \sum_{j=1}^{N-1} \frac{d_{jr} d_{js} d_{-j}^2 + d_{-jr} d_{-js} d_j^2}{|d_j|^5} \\ &= -\frac{3N}{8} \sum_{j=1}^{N-1} \frac{\sin(\pi jr/N) \sin \pi j(r-2)/N}{\sin^3 \pi j/N}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 U_1}{\partial y_r \partial y_s} &= -\frac{3N}{8} \sum_{j=1}^{N-1} \frac{d_{jr} d_{js} d_{-j}^2 + d_{-jr} d_{-js} d_j^2}{|d_j|^5} \\ &= \frac{3N}{8} \sum_{j=1}^{N-1} \frac{\sin(\pi jr/N) \sin \pi j(r-2)/N}{\sin^3 \pi j/N}. \end{aligned}$$

Since  $U_1$  does not depend on  $z_N$ , the nonzero partial derivatives involving  $x_N$  and  $y_N$  come only from  $U_2$ , which is

$$U_2 = \sum_{j=0}^{N-1} \frac{1}{|q_j - q_N|}.$$

The individual distances to the mass at the center are

$$\begin{aligned}
 |q_j - q_N| &= \left| \omega^j + \sum_{k=0}^{N-1} \omega^{jk} z_k + \left(1 + \frac{N}{m}\right) z_N \right| \\
 &= \left[ 1 + \sum_k (\omega^{j(k-1)} z_k + \omega^{j(1-k)} \bar{z}_k) + \sum_{k,l} \omega^{j(k-l)} z_k \bar{z}_l \right. \\
 &\quad \left. + \left(1 + \frac{N}{m}\right) \left( \omega^{-j} z_N + \sum_k \omega^{-jk} \bar{z}_k z_N + \omega^j \bar{z}_N + \sum_k \omega^{jk} z_k \bar{z}_N \right) \right. \\
 &\quad \left. + \left(1 + \frac{N}{m}\right)^2 z_N \bar{z}_N \right]^{1/2}.
 \end{aligned}$$

The computations of the second-order derivatives are slightly simpler than those for  $U_1$ , since for  $z = 0$  we have  $|q_j - q_N| = 1$ , and with it we find

$$\begin{aligned}
 \frac{\partial^2 U_2}{\partial x_r^2} &= \frac{N}{2}, \quad \frac{\partial^2 U_2}{\partial x_r \partial x_s} = \frac{3N}{2}, \quad \frac{\partial^2 U_2}{\partial x_2 \partial x_N} = \frac{3N}{2} \left(1 + \frac{N}{m}\right), \\
 \frac{\partial^2 U_2}{\partial x_N^2} &= \frac{N}{2} \left(1 + \frac{N}{m}\right)^2, \quad \frac{\partial^2 U_2}{\partial y_r^2} = \frac{N}{2}, \quad \frac{\partial^2 U_2}{\partial y_r \partial y_s} = -\frac{3N}{2}, \\
 \frac{\partial^2 U_2}{\partial y_2 \partial y_N} &= -\frac{3N}{2} \left(1 + \frac{N}{m}\right), \quad \frac{\partial^2 U_2}{\partial y_N^2} = \frac{N}{2} \left(1 + \frac{N}{m}\right)^2.
 \end{aligned}$$

The Hessian of  $U$  with respect to  $(x_1, \dots, x_N, y_1, \dots, y_N)$  therefore has the following relatively simple form

$$\Delta = \begin{pmatrix} B+C & 0 \\ 0 & B-C \end{pmatrix}, \tag{20}$$

with the diagonal matrix

$$B = \begin{pmatrix} b_1 & & & & \\ & b_2 & & & \\ & & b_3 & & \\ & & & \ddots & \\ & & & & b_{N-1} \\ & & & & & b_N \end{pmatrix}$$

and the cross-diagonal matrix

$$C = C^T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & c_2 \\ 0 & 0 & 0 & \cdots & c_3 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & c_{N-1} & \cdots & 0 & 0 \\ 0 & c_N & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The nonzero entries were found above and are

$$\begin{aligned}
b_r &= \frac{N}{8} \sum_{j=1}^{N-1} \frac{\sin^2 \pi jr/N}{\sin^3 \pi j/N} + \frac{mN}{2}, \quad 1 \leq r \leq N-1, \\
b_N &= \frac{mN}{2} \left(1 + \frac{N}{m}\right)^2, \\
c_r &= -\frac{3N}{8} \sum_{j=1}^{N-1} \frac{\sin \pi jr/N \sin \pi(r-2)j/N}{\sin^3 \pi j/N} + \frac{3mN}{2}, \\
c_2 = c_N &= \frac{3mN}{2} \left(1 + \frac{N}{m}\right), \quad 3 \leq r \leq N-1.
\end{aligned}$$

The moment of inertia is given by

$$\begin{aligned}
I &= \frac{1}{2} \sum_{j=0}^{N-1} |q_j|^2 + \frac{m}{2} |q_N|^2 \\
&= \frac{N}{2} \left(1 + z_1 + \bar{z}_1 + \sum_{j=0}^N |z_j|^2 + \frac{N}{m} |z_N|^2\right). \tag{21}
\end{aligned}$$

The manifold  $I = I_0 = N/2$  is therefore in its first approximation determined by

$$\frac{1}{2}(z_1 + \bar{z}_1) = \operatorname{Re} z_1 = 0.$$

By imposing  $z_1$  to be real, we select a representative from the rotational equivalence classes. Setting  $z_0 = 0$ ,  $\operatorname{Im} z_1 = 0$ , and  $\operatorname{Re} z_1 = x_1$ , we find that  $\partial I / \partial x_1|_{z=0} = N$ , so that we can solve  $I - I_0 = 0$  for  $x_1$  as a function of the remaining variables, that is,

$$x_1 = \phi(z_2, \dots, z_N).$$

Changing variables by

$$\begin{aligned}
x'_1 &= x_1 - \phi(z_2, \dots, z_N), \\
z'_2 &= z_2, \\
&\vdots \\
z'_N &= z_N
\end{aligned}$$

brings the manifold  $I = I_0$  locally into the hyperplane  $x'_1 = 0$ . Thus  $z'_2, \dots, z'_N$  are valid local coordinates for a representative of an equivalence class. The Hessian for it will give us the desired information. Thus we need the derivatives with respect to  $x'_2, \dots, x'_N, y'_2, \dots, y'_N$ , and they are

$$\begin{aligned}
\frac{\partial U}{\partial x'_r} &= \frac{\partial U}{\partial x_r} + \frac{\partial U}{\partial x_1} \frac{\partial \Phi}{\partial x_r}, \quad 2 \leq r \leq N, \\
\frac{\partial^2 U}{\partial x'_r \partial x'_s} &= \frac{\partial}{\partial x'_s} \left( \frac{\partial U}{\partial x_r} + \frac{\partial U}{\partial x_1} \frac{\partial \Phi}{\partial x_r} \right) + \frac{\partial}{\partial x_1} \left( \frac{\partial U}{\partial x_r} + \frac{\partial U}{\partial x_1} \frac{\partial \Phi}{\partial x_r} \right) \frac{\partial \Phi}{\partial x_s}.
\end{aligned}$$

Fortunately all first-order and most second-order derivatives of  $\Phi$  are zero for  $z'_2 = \dots = z'_n = 0$ . The only nonzero ones, as found from (21), are

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial x_r'^2} &= \frac{\partial^2 \Phi}{\partial y_r'^2} = -1, & 2 \leq r \leq N, \\ \frac{\partial^2 \Phi}{\partial x_N'^2} &= \frac{\partial^2 \Phi}{\partial y_N'^2} = -\left(1 + \frac{N}{m}\right).\end{aligned}$$

It shows that only the terms on the diagonal of the Hessian in (20) get modified. For this we also need

$$\frac{\partial U}{\partial x_1}|_{z'=0} = -\frac{N}{4} \sum_{j=1}^{N-1} \frac{1}{\sin \pi j/N} - mN.$$

Combining all of this we see that the new Hessian  $\Delta$  has the same form as in (20), except that the matrices  $B$  and  $C$  are now of size  $(N-1) \times (N-1)$  and of the form

$$B = \begin{pmatrix} b_2 & & & \\ & b_3 & & \\ & & \ddots & \\ & & & b_{N-1} \\ & & & & b_N \end{pmatrix}, \quad C = \begin{pmatrix} & & & c_2 \\ & & & c_3 \\ & & \ddots & \\ & & & c_{N-1} \\ c_N & & & \end{pmatrix}.$$

In order to simplify the notation we introduce the following abbreviations for sums that have appeared already:

$$\begin{aligned}S &= \frac{1}{2} \sum_{j=1}^{N-1} \frac{1}{\sin \pi j/N}, \\ R_r &= \frac{1}{4} \sum_{j=1}^{N-1} \frac{\sin^2 \pi r j/N}{\sin^3 \pi j/N} + S, \\ T_r &= \frac{3}{4} \sum_{j=1}^{N-1} \frac{\sin(\pi r j/N) \sin \pi j(r-2)/N}{\sin^3 \pi j/N}.\end{aligned}$$

With it we can write the entries for our new Hessian as

$$\begin{aligned}b_r &= \frac{N}{2} [R_r + 3m], & 2 \leq r \leq N-1, \\ b_N &= \frac{N}{2} \left(1 + \frac{N}{m}\right) [3m + N + S], \\ c_r &= \frac{N}{2} [-T_r + 3m], & 3 \leq r \leq N-1, \\ c_2 = c_3 &= \frac{3mN}{2} \left(1 + \frac{N}{m}\right).\end{aligned}$$

The Hessian  $\Delta$  becomes singular when  $B \pm C$  is singular. Due to the structures of  $B$  and  $C$ , this can be learned from  $2 \times 2$  submatrices or from a single term. The former is seen from the submatrices

$$D_{\pm}(N, r) = \begin{pmatrix} b_r & \pm c_r \\ \pm c_r & b_s \end{pmatrix}$$

with  $2 \leq r < N/2 + 1$  and  $s = N + 2 - r$ . When  $N$  is even, the two diagonals cross in one position at an element with index  $N/2 + 1$ . Also denote these  $1 \times 1$  submatrices by  $D_{\pm}(N, r)$ , where  $2r = N + 2$  now.

In most cases the determinants of  $D_{\pm}(N, r)$  are singular for a unique value of  $m$ , which we will denote by  $m(N, r)$ . One exception is when  $N$  is even and the two diagonals cross. In this case only  $D_+(N, N/2 + 1)$  contains  $m$  and we obtain

$$m(N, r) = \frac{T_r - R_r}{6}, \quad 2r = N + 2. \quad (22)$$

For  $3 \leq r < N/2 + 1$  with  $s = N + 2 - r$ , the corresponding value is

$$m(N, r) = \frac{T_s^2 - R_r R_s}{3(R_r + R_s + 2T_r)}.$$

It is easily seen that this formula is also valid for (22). For  $r = 2$  the formula is different due to the occurrence of the factor  $1 + N/m$ . Nevertheless, here too we find a unique value for  $m$ , where the two submatrices are singular. It is given by

$$m(N, 2) = \frac{R_2(N + S)}{3(2N - R_2 - S)}.$$

For  $N = 3$  and  $4$  these values have to agree with those obtained previously in sections 4 and 5. We leave this verification to the reader.

For larger  $N$  it becomes impractical to evaluate these formulas by hand. For this reason we give the values generated by machine. Figure 4 also shows which masses are positive and thus meaningful for the  $N$ -body problem.

For a central mass near a critical value, that is,  $m = m(N, k) + \varepsilon$ , we can then look for new relative equilibria, which bifurcate from the regular  $N$ -gon for  $\varepsilon$  near zero. The Hopf bifurcation theorem and the symmetry of  $U$  can be used to show the existence of these new relative equilibria. Unfortunately, this result is rather weak, since the bifurcation could be degenerate. For this reason it is necessary to normalize the higher order terms in  $U$  in order to see that the bifurcation really occurs. This work was carried out in [4], and we refer the reader there for the details.

## APPENDIX A: The Area of a Triangle

Consider two triangles with the vertices

$$(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2)$$

| $N$ | $k = 2$  | 3      | 4     | 5      | 6      | 7      | 8      | 9      | 10     |
|-----|----------|--------|-------|--------|--------|--------|--------|--------|--------|
| 2   | 0.138    |        |       |        |        |        |        |        |        |
| 3   | 0.770    |        |       |        |        |        |        |        |        |
| 4   | 2.379    | -0.250 |       |        |        |        |        |        |        |
| 5   | 6.478    | -0.244 |       |        |        |        |        |        |        |
| 6   | 20.906   | -0.220 | 0.006 |        |        |        |        |        |        |
| 7   | -643.284 | -0.181 | 0.324 |        |        |        |        |        |        |
| 8   | -37.929  | -0.130 | 0.698 | 0.996  |        |        |        |        |        |
| 9   | -25.443  | -0.069 | 1.119 | 1.773  |        |        |        |        |        |
| 10  | -21.723  | 0.001  | 1.581 | 2.641  | 3.012  |        |        |        |        |
| 11  | -20.270  | 0.079  | 2.079 | 3.588  | 4.391  |        |        |        |        |
| 12  | -19.738  | 0.165  | 2.610 | 4.605  | 5.893  | 6.337  |        |        |        |
| 13  | -19.671  | 0.258  | 3.172 | 5.686  | 7.506  | 8.457  |        |        |        |
| 14  | -19.872  | 0.357  | 3.760 | 6.825  | 9.220  | 10.734 | 11.251 |        |        |
| 15  | -20.242  | 0.462  | 4.374 | 8.018  | 11.025 | 13.152 | 14.251 |        |        |
| 16  | -20.726  | 0.572  | 5.011 | 9.261  | 12.916 | 15.701 | 17.440 | 18.030 |        |
| 17  | -21.290  | 0.687  | 5.671 | 10.551 | 14.885 | 18.371 | 20.802 | 22.048 |        |
| 18  | -21.913  | 0.807  | 6.351 | 11.883 | 16.929 | 21.153 | 24.325 | 26.287 | 26.950 |
| 19  | -22.581  | 0.931  | 7.050 | 13.257 | 19.041 | 24.041 | 27.997 | 30.730 | 32.124 |

**Fig. 4.** Critical values  $m(N, k)$  of the central mass in the regular  $N$ -gon.

and

$$(x'_0, y'_0), \quad (x'_1, y'_1), \quad (x'_2, y'_2).$$

Their areas  $A$  and  $A'$  with appropriate signs are given by

$$2A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{and} \quad 2A' = \begin{vmatrix} x'_1 & y'_1 & 1 \\ x'_2 & y'_2 & 1 \\ x'_3 & y'_3 & 1 \end{vmatrix}.$$

The determinants can be rewritten as

$$-2A = \begin{vmatrix} 0 & 0 & 0 & 1 \\ x_1 & y_1 & 1 & 0 \\ x_2 & y_2 & 1 & 0 \\ x_3 & y_3 & 1 & 0 \end{vmatrix} \quad \text{and} \quad 2A' = \begin{vmatrix} 0 & 0 & 1 & 0 \\ x'_1 & y'_1 & 0 & 1 \\ x'_2 & y'_2 & 0 & 1 \\ x'_3 & y'_3 & 0 & 1 \end{vmatrix}.$$

By multiplying these two determinants row by row, one obtains

$$-4AA' = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & x_1x'_1 + y_1y'_1 & x_1x'_2 + y_1y'_2 & x_1x'_3 + y_1y'_3 \\ 1 & x_2x'_1 + y_2y'_1 & x_2x'_2 + y_2y'_2 & x_2x'_3 + y_2y'_3 \\ 1 & x_3x'_1 + y_3y'_1 & x_3x'_2 + y_3y'_2 & x_3x'_3 + y_3y'_3 \end{vmatrix}.$$

If one sets

$$r_{ij}^2 = (x_i - x'_j)^2 + (y_i - y'_j)^2$$

for the distance between points in the different triangles, one can write

$$x_ix'_j + y_iy'_j = -\frac{1}{2}r_{ij}^2 + \frac{1}{2}(x_i^2 + y_i^2) + \frac{1}{2}(x_j'^2 + y_j'^2).$$

In the determinant for  $-4AA'$ , subtract from the  $i$ th row ( $i = 2, 3, 4$ ) the first row after multiplying it with  $\frac{1}{2}(x_i^2 + y_i^2)$ , and subtract from the  $j$ th column

( $j = 2, 3, 4$ ) the first column after multiplying it with  $\frac{1}{2}(x_i'^2 + y_i'^2)$ . One obtains

$$-4AA' = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -r_{11}^2/2 & -r_{12}^2/2 & -r_{13}^2/2 \\ 1 & -r_{21}^2/2 & -r_{22}^2/2 & -r_{23}^2/2 \\ 1 & -r_{31}^2/2 & -r_{32}^2/2 & -r_{33}^2/2 \end{vmatrix}.$$

Finally, multiply the last three rows with  $-2$  and the first column with  $-1/2$  so that

$$-16AA' = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & r_{11}^2 & r_{12}^2 & r_{13}^2 \\ 1 & r_{21}^2 & r_{22}^2 & r_{23}^2 \\ 1 & r_{31}^2 & r_{32}^2 & r_{33}^2 \end{vmatrix}. \quad (23)$$

When the two triangles coincide, that is, when

$$(x_i, y_i) = (x_i', y_i'), \quad i = 1, 2, 3,$$

we obtain the desired formula for the area of a triangle:

$$-16A^2 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 \end{vmatrix}.$$

By expanding this determinant the standard formula

$$16A^2 = (r_{12} + r_{23} + r_{13})(-r_{12} + r_{23} + r_{13})(r_{12} - r_{23} + r_{13})(r_{12} + r_{23} - r_{13})$$

for the area of a triangle is found, which is typically given in the form

$$A^2 = s(s - r_{12})(s - r_{23})(s - r_{13})$$

with

$$s = (r_{12} + r_{23} + r_{13})/2.$$

Of importance in this derivation is that the ideas can be extended to higher dimensional objects as long as the factors are adjusted accordingly. For example, the volume  $V$  of a tetrahedron is given by  $288V^2 = F$ , where

$$F = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix}.$$

Differentiating this determinant, for example, with respect to  $r_{12}^2$ , leads to

$$\frac{\partial F}{\partial r_{12}^2} = -2 \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & r_{12}^2 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{34}^2 & 0 \end{vmatrix}.$$

In accordance with (23) this determinant can be interpreted as the product of the area of two triangles containing the points  $(2, 3, 4)$  and  $(1, 3, 4)$ , respectively. Denoting by  $A_i$  the triangle where the  $i$ th point is omitted, we find in general that

$$\frac{\partial F}{\partial r_{ij}^2} = 32A_iA_j.$$

One point of caution has to be made. The way in which formula (23) was used requires that the two triangles lie in the same plane. In other words, the above formula is only valid when the volume of the tetrahedron is zero.

## APPENDIX B: *Mathematica* Code for the Four-Body Problem

```

m = { 1, 1, 1, mc + Epsilon };
z = {Lambda, Sigma, r[1, 2], r[1, 3], r[1, 4],
      r[2, 3], r[2, 4], r[3, 4]};

rplz0 = {Lambda -> Lambda0, Sigma -> Sigma0,
         r[1, 2] -> Sqrt[3], r[1, 3] -> Sqrt[3], r[1, 4] -> 1,
         r[2, 3] -> Sqrt[3], r[2, 4] -> 1,
         r[3, 4] -> 1 };

F[r12_, r13_, r14_, r23_, r24_, r34_] :=
  Det[{{0, 1, 1, 1, 1},
        {1, 0, r12^2, r13^2, r14^2},
        {1, r12^2, 0, r23^2, r24^2},
        {1, r13^2, r23^2, 0, r34^2},
        {1, r14^2, r24^2, r34^2, 0}}] ;
Inrt = 1/2*Sum[Sum[ m[[i]]*m[[j]]*r[i, j]^2,{i,1,j-1}},{j,2,4}] ;
Inrt0 = Inrt /. rplz0 ;
U = Sum[Sum[m[[i]]*m[[j]]/r[i, j], {i,1,j-1}], {j,2,4}] ;
F = F[r[1, 2], r[1, 3], r[1, 4], r[2, 3], r[2, 4], r[3, 4]] ;
V = U + Lambda * (Inrt - Inrt0) + Sigma*F/2;
Do[eq[i] = D[ V, z[[i]] ], {i, 1, 8}];

```

Obtain  $\lambda$  and  $\sigma$  at order 0, and determine critical mass value:

```

Do [eq0[i] = Simplify[eq[i] /. rplz0], {i, 1, 8}];
rplLam0Sig0 =
  Part[Solve[{eq0[5] == 0, eq0[6] == 0}, {Lambda0, Sigma0}], 1]
Wronskian =
  Simplify[Det[
    Table[D[V, z[[i]]], z[[j]]] /. rplz0 /.
      rplLam0Sig0 /. Epsilon -> 0, {i, 1, 8}, {j, 1, 8}]]]
rplmc = Part[Solve[Wronskian == 0, mc], 4]
rplz = Table[
  z[[i]] -> Simplify[z[[i]] /. rplz0 /. rplLam0Sig0 /. rplmc]

```

```

+ Epsilon*b[i] + Epsilon^2 *c[i], {i, 1, 8}]
Do[eq1[i] = Series[eq[i] /. rplz /. rplmc, {Epsilon,0,2}],{i,1,8}]

```

First-order terms in  $\epsilon$ :

```

Do[eq1[i] = FullSimplify[Coefficient[eq1[i], Epsilon, 1]],{i,1,8}]
eqs = Table[eq1[i] == 0, {i, 1, 6}];
ord1 = FullSimplify[Part[Solve[eqs, Array[b, 6]], 1]]
Do[Print["eq1[" , i, "]=", FullSimplify[eq1[i] /. ord1]],{i,7,8}]

```

Second-order terms with bifurcation equations:

```

Do[eq2[i] = FullSimplify[Coefficient[eq1[i], Epsilon, 2] /. ord1],
  {i, 1, 8}];
eqs = Table[eq2[i] == 0, {i, 1, 6}];
ord2 = Simplify[Part[Solve[eqs, Array[c, 6]], 1]]
bifeq1 = Factor[FullSimplify[eq2[7] /. ord2]]
bifeq2 = Factor[FullSimplify[eq2[8] /. ord2]]
bif = FullSimplify[Solve[{bifeq1 == 0, bifeq2 == 0}, {b[7], b[8]}]]

```

Determine distances for one of the new configurations:

```

rplrij = Drop[rplz, 2];
Do[rplrij = rplrij /. c[j] -> 0, {j, 3, 8}]
rplrij = FullSimplify[rplrij /. ord1 /. Part[bif, 3]]

```

## APPENDIX C: *Mathematica* Code for the Planar Five-Body Problem

The code is very similar to the one for the four-body problem. The major difference is the additional constraints, which insure that the configuration can be realized in the plane. The *Mathematica* code for central configurations in  $\mathbf{R}^3$  is not listed here, but it is very similar to the one shown. The difference is that only one constraint is needed, which insures that the configuration can be realized in  $\mathbf{R}^3$ . This constraint is given by a  $6 \times 6$  determinant, which is the obvious extension of the one listed in the program.

```

m={ 1, 1, 1, 1, mc + Delta*Epsilon^2 };
rplz0={Lambda -> Lambda0,
  Sigma1 -> Sigma10, Sigma2 -> Sigma20, Sigma3 -> Sigma30,
  r[1, 2] -> Sqrt[2],r[1,3] -> 2,r[1,4] -> Sqrt[2],r[1,5] -> 1,
  r[2, 3] -> Sqrt[2],r[2,4] -> 2,r[2,5] -> 1,
  r[3, 4] -> Sqrt[2],r[3,5] -> 1,
  r[4, 5] -> 1};

z={Lambda, Sigma1, Sigma2, Sigma3, r[1,2], r[1,3], r[1,4], r[1,5],
  r[2,3], r[2,4], r[2,5], r[3,4], r[3,5], r[4,5]};
F[r12_, r13_, r14_, r23_, r24_, r34_] :=
  Det[{{0, 1, 1, 1, 1},
    {1, 0, r12^2, r13^2, r14^2},

```

```

{1, r12^2, 0, r23^2, r24^2},
{1, r13^2, r23^2, 0, r34^2},
{1, r14^2, r24^2, r34^2, 0}}] ;
Inrt = 1/2*Sum[Sum[ m[[i]]*m[[j]]*r[i, j]^2,{i,1,j-1}},{j,2,5}] ;
Inrt0 = Inrt/. rplz0 ;
U = Sum[Sum[m[[i]]*m[[j]]/r[i, j], {i,1,j-1}], {j,2,5}] ;
F1 = F[r[1, 2], r[1, 3], r[1, 4], r[2, 3], r[2, 4], r[3, 4]] ;
F2 = F[r[1, 2], r[1, 3], r[1, 5], r[2, 3], r[2, 5], r[3, 5]] ;
F3 = F[r[2, 3], r[2, 4], r[2, 5], r[3, 4], r[3, 5], r[4, 5]] ;
V = U+Lambda*(Inrt-Inrt0)+(Sigma1*F1+Sigma2*F2+Sigma3*F3)/2;
Do[eq[i] = D[ V, z[[i]] ], {i, 1, 14}];

```

Obtain Lambdas and Sigmas at order 0, and determine critical mass value:

```

Do[ eq0[i] = Simplify[eq[i] /. rplz0 ], {i, 1, 14}]
rplLam0Sig0 =
  FullSimplify[ Part[ Solve[
    Table[eq0[i] = 0, {i,1,14}], {Lambda0,Sigma10,Sigma20,Sigma30}],1]]
Wronskian =
  Simplify[Det[Table[Simplify[
    D[V, z[[i]], z[[j]]] /. rplz0 /. rplLam0Sig0 /. Epsilon -> 0],
    {i, 1, 14}, {j, 1, 14}]]]
rplmc = Last[Solve[Wronskian == 0, mc]]
rplz = Table[
  z[[i]] -> Simplify[z[[i]] /. rplz0 /. rplLam0Sig0 /. rplmc]+
  Epsilon*b[i]+Epsilon^2*c[i]+Epsilon^3*d[i],{i,1,14}]
Do[equ[i] = Series[ eq[i] /. rplz /. rplmc, {Epsilon,0,3}], {i,1,14}]

```

Solve equations at first order. One needs to solve linear systems of equations in two stages, as *Mathematica* is not very good at solving larger systems. It does not simplify intermediate results on its own.

```

Do[eq1[i] = FullSimplify[Coefficient[equ[i],Epsilon,1]],{i,1,14}]
rplb8to12 =
  FullSimplify[Part[Solve[Array[eq1, 5]=0,Table[b[i],{i,8,12}]],1]]
eq1a6to14 = FullSimplify[Table[eq1[i] /. rplb8to12, {i, 6, 14}]]
rplb1to7 =
  FullSimplify[
    Part[Solve[Table[eq1a6to14[[i]] = 0, {i,1,9}], Array[b,7]],1]]
ord1 = Join[rplb1to7, FullSimplify[rplb8to12 /. rplb1to7]]

```

Check that the remaining two equations are satisfied:

```
Do[Print["eq1[" , i, "]=", FullSimplify[eq1[i] /. ord1]], {i,13,14}]
```

Solve equations at second order:

```

Do[eq2[i] =
  FullSimplify[Coefficient[equ[i], Epsilon, 2] /. ord1],{i,1,14}];
rplb8to12 =
  FullSimplify[Part[Solve[Array[eq2,5]=0,Table[c[i],{i,8,12}]],1]];

```

```
eq1a6to14 = FullSimplify[Table[eq2[i] /. rplb8to12, {i, 6, 14}]];
rplb1to7 =
  FullSimplify[
    Part[Solve[Table[eq1a6to14[[i]] = 0, {i, 1, 9}], Array[c, 7]], 1]];
ord2 = Join[rplb1to7, FullSimplify[rplb8to12 /. rplb1to7]];
Do[Print["eq2[" , i, "]" = " , FullSimplify[eq2[i] /. ord2]], {i, 13, 14}]
```

Solve equations at third order:

```
Do[eq3[i] =
  FullSimplify[Coefficient[equ[i], Epsilon, 3] /. ord2 /. ord1], {i, 1, 14}];
rplb8to12 =
  FullSimplify[Part[Solve[Array[eq3, 5] = 0, Table[d[i], {i, 8, 12}]], 1]];
eq1a6to14 = FullSimplify[Table[eq3[i] /. rplb8to12, {i, 6, 14}]];
rplb1to7 =
  FullSimplify[
    Part[Solve[Table[eq1a6to14[[i]] = 0, {i, 1, 7}], Array[d, 7]], 1]];
ord3 = Join[rplb1to7, FullSimplify[rplb8to12 /. rplb1to7]]

(* Solve bifurcation equations *)

bifeq1 = FullSimplify[eq3[13] /. ord3]
bifeq2 = FullSimplify[eq3[14] /. ord3]
rplbif = Solve[{bifeq1 == 0, bifeq2 == 0}, {b[13], b[14]}]
```

Choose Delta = +1 or -1 and, depending on that, one or the other of the solutions to the bifurcation equations.

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