THE LUNAR THEORY OF HILL AND BROWN*

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Abstract. Hill and Brown solved the equations of motions of the Moon with the help of multiple Fourier series. We describe their method and show how it can be adapted, so that all work can be carried out by a computer with the help of an algebraic processor called POLYPAK.

1. Introduction

It is customary to proceed in two steps when one tries to find an analytic solution to the motion of the Moon. In the first step the so-called main problem of lunar theory is solved. It consists of a simplified three body problem under the assumptions that two bodies, Earth and Moon, stay always close together and that the Sun describes an elliptic orbit around their center of mass. In a second step all other influences on the motion of the moon are considered. These include among others, the oblateness of the Earth and the perturbations by the planets.

The most accurate analytic solutions to the main problem are the one of Delaunay as corrected by Deprit *et al.* (1971) and the one of Brown (1899–1908). Brown's work has been checked and improved in parts with the help of differential corrections. If further improvements are to be made it can most likely be accomplished by a revision of the planetary perturbations. Since a complete revision of Brown's work has not yet been accomplished we feel it is necessary to redo his computations for the main problem before proceeding to the planetary perturbations. As the computations are done by machine this will give a reliable solution in which the latest numerical values for the parameters can be used.

All computer programs are again written in the higher level language PL/I. They are called as subroutines by POLYPAK, which is our package for the manipulation of real or complex power series in several variables.

In an earlier paper we discussed our way for finding the zero and first order solution to the main problem. The second and higher order terms are determined differently, and we will describe in this paper how we computed them.

2. Notation and Equations of Motion

The following notation has been adopted from Brown.

n, n' the observed mean angular velocity of the Moon and the Sun.

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a', e' mean distance and eccentricity for the Sun's orbit.

$\kappa = (E+M)/(n-n')^2$ where E and M are the masses of the Earth and Moon.

$m = n'/(n-n') = 0.08084\,89338\,08312$. Because the solution converges very slowly when m is kept as a formal parameter Brown and Hill used this numerical value from the outset.

(x, y, z) coordinates of the Moon in a system which is Earth centered and which rotates around its z-axis with constant angular velocity n' so that the positive x-axis always points in the direction of the mean position of the Sun.

$u = x+iy, v = x-iy$ complex position coordinates for the x, y-plane.

$r = (x^2+y^2+z^2)^{1/2} = (uv+z^2)^{1/2}$ distance between Earth and Moon.

$\xi = \exp i(n-n')(t-t_0)$ is used as the independent variable instead of the time t. As the solution is a Poisson series in t it will become a power series in ξ.

$D = \xi(\mathrm{d}/\mathrm{d}\xi)$ differential operator which is related to the derivative with respect to time by $\mathrm{d}/\mathrm{d}t = i(n-n')D$.

The equations of motion are

$$\begin{aligned} D^2u + 2m\,Du + \tfrac{3}{2}m^2(u+v) - \frac{\kappa u}{r^3} &= -\frac{\partial\Omega}{\partial v}, \\ D^2v - 2m\,Dv + \tfrac{3}{2}m^2(u+v) - \frac{\kappa v}{r^3} &= -\frac{\partial\Omega}{\partial u}, \\ D^2z \qquad\quad - m^2z \qquad\quad - \frac{\kappa z}{r^3} &= -\frac{1}{2}\frac{\partial\Omega}{\partial z}. \end{aligned} \tag{1}$$

For $\Omega = 0$ the problem is known as Hill's lunar problem. Therefore Ω describes the difference to the main problem. The exact form for Ω is not needed for the following discussion. It is sufficient to note that it depends on the position of the Moon, the parameters of the Sun's orbit and on the time with period $2\pi/n'$. We thus have

$$\Omega = \Omega(u, v, z, a', e', \xi^m). \tag{2}$$

We can expand $a'^2\Omega$ into homogeneous polynomials in the variables u/a', v/a', z/a' starting with terms of second order. The coefficients will be functions of e' and ξ^m only.

3. Hill's Lunar Problem

The zero and first order solutions to Hill's Lunar problem serve as a starting point to the solution for the main problem. These solutions are found differently (Schmidt,

1979) than the higher order approximations. They have the following form

$$u = a\xi(u_0 + u_e)\,,$$
$$z = -iaz_1\,,$$

with

$$u_0 = \sum a_{2j}\xi^{2j}\,,$$
$$u_e = e\sum(\varepsilon_{2j}\xi^{2j+c_0} + \varepsilon'_{2j}\xi^{2j-c_0})\,,$$
$$z_1 = k\sum k_{2j}(\xi^{2j+g_0} - \xi^{-2j-g_0})\,.$$

Although the summation should be over all integers j, the absolute value of the real coefficients a_{2j}, ε_{2j}, ε'_{2j} and k_{2j} drop off very quickly when j is away from zero. If the level of truncation is set at 10^{-15}, then $2j$ has to be at most ± 12. The real numbers c_0 and g_0 describe the motion of the perigee and node respectively. The arbitrary parameters a, e and k are closely related to the mean distance, the eccentricity and the inclination of the Moon's orbit. They are not identical to these values because the following choices have been made to simplify the computations

$$a_0 = 1, \qquad \varepsilon_0 - \varepsilon'_0 = 1, \qquad k_0 = 1\,. \tag{3}$$

If a comparison with Delaunay's result is desired additional work will have to be performed at a later stage. Since we want to compare our results with those of Brown we will adopt his conventions here.

4. Higher Order Approximations

With the introduction of the additional parameter $\alpha = a/a'$ and by considering Equation (2) we can write the solution to Equation (1) as a power series in several variables in the form

$$\xi^{-1}u = a\sum a_{\lambda\sigma}e^{\lambda_1}e'^{\lambda_2}k^{\lambda_3}\alpha^{\lambda_4}\xi^{c\sigma_1 + m\sigma_2 + g\sigma_3 + \sigma_4}\,.$$

The series for ξv is conjugate to the one above and the series for iz has the same form.

The quadruple $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is called the characteristic and $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ the order of a term. Summation is over all nonnegative λ's. Due to d'Alembert's characteristic certain coefficients will turn out to be zero, and the restriction $|\sigma_k| \leq \lambda_k$, $k = 1$, 2 and 3, holds for the summation over the σ's. σ_4 ranges over all even or all odd integers depending on the characteristic λ. The series c and g start with the terms c_0 and g_0 respectively. They are power series in the squares of the four formal parameters e, e', k and α. Their coefficients have to be found along with the coefficients $a_{\lambda\sigma}$.

We denote all terms with characteristic λ in $\xi^{-1}u/a$ by u_λ and those in iz/a by z_λ.

Assuming that all terms of order less than $|\lambda|$ are known we write

$$\xi^{-1}u = a\Big(\sum_{|\mu|<|\lambda|} u_\mu + u_\lambda\Big),$$

$$iz = a\Big(\sum_{|\mu|<|\lambda|} z_\mu + z_\lambda\Big).$$

From Equation (1) we then obtain the following linear differential equation for u_λ

$$\begin{aligned}(D+m+1)^2 u_\lambda + M(\xi)u_\lambda + N(\xi)v_\lambda &= \\ = \operatorname{char}_\lambda \Big\{ -\xi^{-1}\frac{\partial \Omega}{\partial v} &+ (\textstyle\sum u_\mu)((\sum u_\mu)(\sum v_\mu) - (\sum z_\mu)^2)^{-3/2} \\ &- (D+m+1)^2 \textstyle\sum u_\mu \Big\}.\end{aligned} \tag{4}$$

A conjugate complex equation holds for v_λ. The series M and N are given by

$$M(\xi) = \sum M_{2j}\xi^{2j} = \tfrac{1}{2}(m^2 + (u_0 v_0)^{-3/2}),$$
$$N(\xi) = \sum N_{2j}\xi^{3j} = \tfrac{3}{2}(m^2\xi^{-2} + u_0^{-1/2} v_0^{-5/2}).$$

The right-hand side of Equation (4) depends only on known terms except for an occasional new term for the series c at orders 3, 5, etc. These new terms will come up because of the last expression in Equation (4) which takes into account that c and g are power series by themselves.

The second term in the right-hand side of Equation (4) arises from the expansion of $\kappa u/r^3$ when only known terms of u are taken into account. This expansion proved to be very time consuming for Brown. It forced him to switch to different methods in the course of his computations. When the computations are performed by machine this expansion provides no difficulty at all. Therefore we will describe now the method which Brown used initially and which we will use to find terms of any order.

The first step of our program consists of evaluating the right-hand side of Equation (4) in order to get all terms of order $|\lambda|$. Since the differential equations are linear we can treat each characteristic λ separately. Among the terms with a specific characteristic λ we find those which are multiplied by $\xi^{\pm\tau+k}$, where τ is one of the values $\sigma_1 c + \sigma_2 m + \sigma_3 g$ and $k = \sigma_4$. If these terms are collected into the series

$$A_\lambda = a\Big(\sum_k A_k \xi^{\tau+k} + A'_k \xi^{-\tau+k}\Big),$$

then the corresponding terms in the solution are

$$u_\lambda = a(\sum b_k \xi^{\tau+k} + b'_k \xi^{-\tau+k}).$$

By inserting this u into Equation (4) and into its conjugate equation we arrive at the following infinite system of linear equations

$$\begin{aligned}(k+\tau_0+1+m)^2 b_k + \sum M_{2j} b_{k-2j} + \sum N_{2j} b'_{2j-k} &= A_k, \\ (k+\tau_0-1-m)^2 b'_{-k} + \sum N_{2j} b_{k+2j} + M_{2j} b'_{-2j-k} &= A'_{-k},\end{aligned} \tag{5}$$

where

$$\tau_0 = \sigma_1 c_0 + \sigma_2 m + \sigma_3 g_0 \quad \text{and} \quad k = 0, \pm 2, \pm 4, \ldots \text{ or } k = \pm 1, \pm 3, \ldots .$$

We illustrate our program with the case where k takes on all even integers. The other case is very similar. We arrange the unknowns in an array x in the following order $x = (\ldots, b'_4, b_{-2}, b'_2, b_0, b'_0, b_2, b'_{-2}, b_4, \ldots)^T$ and the right-hand side in an array y with the same ordering. The coefficient matrix A for the system of linear equations $Ax = y$ reads then

$$A = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & (-1+\tau_0+m)^2+M_0 & N_0 & M_{-2} & N_{-2} & M_{-4} & N_{-4} & \cdots \\ \cdots & N_0 & (-3+\tau_0-m)^2+M_0 & N_2 & M_2 & N_4 & M_4 & \cdots \\ \cdots & M_2 & N_2 & (1+\tau_0+m)^2+M_0 & N_0 & M_{-2} & N_{-2} & \cdots \\ \cdots & N_{-2} & M_{-2} & N_0 & (-1+\tau_0-m)+M_0 & N_2 & M_2 & \cdots \\ \cdots & M_4 & N_4 & M_2 & N_2 & (3+\tau_0+m)^2+M_0 & N_0 & \cdots \\ \cdots & N_{-4} & M_{-4} & N_{-2} & M_{-2} & N_0 & (1+\tau_0-m)^2+M_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

The dominant terms are the 2×2 submatrices along the diagonal. The matrix has band form because the coefficients M_{2j} and N_{2j} diminish rapidly away from the diagonal. Furthermore the matrix is symmetric as $M_{2j} = M_{-2j}$.

The terms on the right-hand side of Equation (5) become negligible when k is away from zero. Thus it is possible to reduce the infinite system $Ax = y$ to a finite one. Our criterion is to keep all terms which are in absolute value above a certain threshold. At the moment it is set at $10^{-16+|\lambda|}$. We also make sure that the reduced coefficient matrix remains symmetric by retaining additional equations if necessary. We then use the method for solving symmetric linear systems of Bunch (1971) and improve the result by iteration. With this approach we try to avoid the problem of small divisors and the loss of accuracy. This had plagued Brown because he could not carry enough decimal digits in his computations by hand.

Certain small divisors still remain. Actually they are zero divisors of the infinite system and they require special treatment.

One case is $\tau = 0$, that is $\sigma_1 = \sigma_2 = \sigma_3 = 0$. The two equations of (5) collapse into one and it is obvious what to do. The other case occurs whenever a new term in the series for c has to be found. This happens when $\tau_0 = \pm c_0$ and $\lambda = (1+2r_1, 2r_2, 2r_3, 2r_4)$. The term to be found will have characteristic $\lambda/e = (2r_1, 2r_2, 2r_3, 2r_4)$ and we will denote its unknown coefficient by $c_{\lambda/e}$. It appears on the right-hand side of Equation (5) always with the same factor. We find

$$A_{2j} = -2(2j + c_0 + m + 1)\varepsilon_{2j} c_{\lambda/e} + B_{2j},$$

$$A'_{-2j} = -2(-2j - c_0 + m + 1)\varepsilon'_{-2j} c_{\lambda/e} + B'_{-2j},$$

where B_{2j} and B'_{-2j} are completely known.

Substitute these values into Equation (5). Multiply the first equation with ε_{2j} the second wtih ε'_{-2j} and sum up over j. In this way Brown (1899, p. 75) is able to determine $c_{\lambda/e}$. It is given by

$$c_{\lambda/e} = \frac{1}{D} \sum B_{2j}\varepsilon_{2j} + B'_{2j}\varepsilon'_{2j}$$

with the denominator

$$D = 2 \sum (2j + c_0 + m + 1)\varepsilon^2_{2j} + (2j - c_0 + m + 1)\varepsilon'^2_{2j} \,.$$

Once $c_{\lambda/e}$ is known and with it the right-hand sides of Equation (5), the system is made nonsingular by replacing one of the equations with another relation between the coefficients. By extending the choice (3) to higher order terms this equation has to be

$$b_0 - b'_0 = 0 \,.$$

The replacement can be accomplished in such a manner that the resulting system remains symmetric so that the system can be solved as before.

5. Results

The previous section gives an outline of our computer program for finding u_λ. The method is basically the one used by Brown for terms through third order. In addition we have to find z_λ. The method is very similar to the one for u_λ and we have omitted the details.

The expression $\xi(\partial\Omega/\partial v)$ in Equation (4) will produce first order terms with characteristic e' and α but none with e or k. The corresponding terms in $u_{e'}$ and u_α are found in the same way as the higher order terms. They have to be added to u_e to give the complete first order solution u_1. After this the higher order terms can be found.

We have computed the solution through sixth order. We kept all terms which were greater than $10^{-16+\text{order}}$ except at order 6 where we were more restrictive. At this point the series would have become too long and many of the terms would have no influence on the final result once the numerical values for the parameters are inserted.

When we compare our results with those of Brown we often disagree in the last or last two digits. Brown does not list all coefficients with the same number of significant digits even within terms with the same characteristic. Inevitably the coefficients which have many nonzero digits are given less accurately than those with many leading zero digits. It is difficult at this point to estimate how much these discrepancies will influence the final result. For this reason we now convert our solution to spherical coordinates and we will evaluate it at the given values of the parameters e, e', k and α. We are then able to compare it to the final result of Brown and to the corrections which have been added to Brown's values over the years. Other comparisons will be made with Delaunay's work as corrected by Deprit *et al.* (1971)

and to the semi-analytical solution which has been developed by Henrard (1978). We will report on our findings in a later paper.

References

Brown, E. W.: 1899, *Mem. Roy. Astron. Soc.* **53**, 39–116, 163–202; 1904, *ibid.* **54**, 1–64; 1905, *ibid.* **57**, 1–145; 1908, *ibid.* **59**, 1–103.
Bunch, J. R. and Parlett, B. N.: 1971, *SIAM J. Numer. Anal.* **8**, 639.
Deprit, A., Henrard, J., and Rom, A.: 1971, *Astron. Astrophys.* **10**, 257.
Henrard, J.: 1978, *S.A.L.E. (Semi-Analytical-Lunar-Ephemeris) Tables,* Report 78/8, Dept. of Math., Namur, Belgium.
Schmidt, D. S.: 1979, *Celest. Mech.* **19**, 279.