philosophical resources then available. When more powerful tools emerge, philosophers seem willing to acknowledge deeper difficulties that would have proved insurmountable for more primitive methods. We may have resolutions which are appropriate to our present level of understanding, but they may appear quite inadequate when we have advanced further. The paradoxes do, after all, go to the very heart of space, time, and motion, and these are profoundly difficult concepts.

Or, has the onion infinitely many layers? If so, we may be faced with an infinite sequence of tasks that does defy completion in a finite time, for the steps become longer, not shorter, as the difficulties become deeper.

BERTRAND RUSSELL

The Problem of Infinity Considered Historically

Zeno's four arguments against motion were intended to exhibit the contradictions that result from supposing that there is such a thing as change, and thus to support the Parmenidean doctrine that reality is unchanging.[1] Unfortunately, we only know his arguments through Aristotle,[2] who stated them in order to refute them. Those philosophers in the present day who have had their doctrines stated by opponents will realise that a just or adequate presentation of Zeno's position is hardly to be expected from Aristotle; but by some care in interpretation it seems possible to reconstruct the so-called "sophisms" which have been "refuted" by every tyro from that day to this.

Zeno's arguments would seem to be "ad hominem"; that is to say, they seem to assume premisses granted by his opponents, and to show that, granting these premisses, it is possible to deduce consequences which his opponents must deny. In order to decide whether they are valid arguments or "sophisms," it is necessary to guess at the tacit premisses,

From *Our Knowledge of the External World* (New York: W. W. Norton & Company, Inc., 1929), lecture 6, pp. 182–198. Reprinted by permission of George Allen & Unwin Ltd. Footnote numbers have been changed to number consecutively throughout the article.

[1] This interpretation is combated by Milhaud, *Les philosophes-géomètres de la Grèce*, p. 140n., but his reasons do not seem to me convincing. All the interpretations in what follows are open to question, but all have the support of reputable authorities.

[2] *Physics* VI. 9, 239b (R.P. 136–139).

and to decide who was the "homo" at whom they were aimed. Some maintain that they were aimed at the Pythagoreans,[3] while others have held that they were intended to refute the atomists.[4] M. Evellin, on the contrary, holds that they constitute a refutation of infinite divisibility,[5] while M. G. Noël, in the interests of Hegel, maintains that the first two arguments refute infinite divisibility, while the next two refute indivisibles.[6] Amid such a bewildering variety of interpretations, we can at least not complain of any restrictions on our liberty of choice.

The historical questions raised by the above-mentioned discussions are no doubt largely insoluble, owing to the very scanty material from which our evidence is derived. The points which seem fairly clear are the following: (1) that, in spite of MM. Milhaud and Paul Tannery, Zeno is anxious to prove that motion is really impossible, and that he desires to prove this because he follows Parmenides in denying plurality;[7] (2) that the third and fourth arguments proceed on the hypothesis of indivisibles, a hypothesis which, whether adopted by the Pythagoreans or not, was certainly much advocated, as may be seen from the treatise *On Indivisible Lines* attributed to Aristotle. As regards the first two arguments, they would seem to be valid on the hypothesis of indivisibles, and also, without this hypothesis, to be such as would be valid if the traditional contradictions in infinite numbers were insoluble, which they are not.

[3] Cf. Gaston Milhaud, *Les philosophes-géomètres de la Grèce*, p. 140n.; Paul Tannery, *Pour l'histoire de la science hellène*, p. 249; John Burnet, *Early Greek Philosophy* [2nd ed. (London: Adam & Charles Black, 1908)], p. 362.

[4] Cf. R. K. Gaye, "On Aristotle, *Physics*, Z ix" [25], esp. 111. Also Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, 1st ed., Vol. 1, 1880, p. 168, who, however, subsequently adopted Paul Tannery's opinion, *Vorlesungen*, 3rd ed. (Vol. 1, p. 200).

[5] "Le mouvement et les partisans des indivisibles," *Revue de Métaphysique et de Morale*, Vol. I [1893], pp. 382–395.

[6] "Le mouvement et les arguments de Zénon d'Elée," *Revue de Métaphysique et de Morale*, Vol. I, pp. 107–125.

[7] Cf. M. Brochard, "Les prétendus sophismes de Zénon d'Elée," *Revue de Métaphysique et de Morale*, Vol. I, pp. 209–215.

We may conclude, therefore, that Zeno's polemic is directed against the view that space and time consist of points and instants; and that as against the view that a finite stretch of space or time consists of a finite number of points and instants, his arguments are not sophisms, but perfectly valid.

The conclusion which Zeno wishes us to draw is that plurality is a delusion, and spaces and times are really indivisible. The other conclusion which is possible, namely, that the number of points and instants is infinite, was not tenable so long as the infinite was infected with contradictions. In a fragment which is not one of the four famous arguments against motion, Zeno says:

"If things are a many, they must be just as many as they are, and neither more nor less. Now, if they are as many as they are, they will be finite in number.

"If things are a many, they will be infinite in number; for there will always be other things between them, and others again between these. And so things are infinite in number."[8]

This argument attempts to prove that, if there are many things, the number of them must be both finite and infinite, which is impossible; hence we are to conclude that there is only one thing. But the weak point in the argument is the phrase: "If they are just as many as they are, they will be finite in number." This phrase is not very clear, but it is plain that it assumes the impossibility of definite infinite numbers. Without this assumption, which is now known to be false, the arguments of Zeno, though they suffice (on certain very reasonable assumptions) to dispel the hypothesis of finite indivisibles, do not suffice to prove that motion and change and plurality are impossible. They are not, however, on any view, mere foolish quibbles: they are serious arguments, raising difficulties which it has taken two thousand years to answer, and which even now are fatal to the teachings of most philosophers.

[8] Simplicius, *Phys.* 140, 28d (R.P. 133); Burnet, *op. cit.*, pp. 364–365.

The first of Zeno's arguments is the argument of the race-course, which is paraphrased by Burnet as follows:[9]

"You cannot get to the end of a race-course. You cannot traverse an infinite number of points in a finite time. You must traverse the half of any given distance before you traverse the whole, and the half of that again before you can traverse it. This goes on *ad infinitum,* so that there are an infinite number of points in any given space, and you cannot touch an infinite number one by one in a finite time."[10]

Zeno appeals here, in the first place, to the fact that any distance, however small, can be halved. From this it follows, of course, that there must be an infinite number of points in a line. But, Aristotle represents him as arguing, you cannot touch an infinite number of points *one by one* in a finite time. The words "one by one" are important.

(1) If *all* the points touched are concerned, then, though you pass through them continuously, you do not touch them "one by one." That is to say, after touching one, there

[9] *Op. cit.*, p. 367.

[10] Aristotle's words are: "The first is the one on the non-existence of motion on the ground that what is moved must always attain the middle point sooner than the end-point, on which we gave our opinion in the earlier part of our discourse." *Phys.* VI. 9, 939b (R.P. 136). Aristotle seems to refer to *Phys.* VI. 2, 223ab [R.P. 136a]: "All space is continuous, for time and space are divided into the same and equal divisions. . . . Wherefore also Zeno's argument is fallacious, that it is impossible to go through an infinite collection or to touch an infinite collection one by one in a finite time. For there are two senses in which the term 'infinite' is applied both to length and to time, and in fact to all continuous things, either in regard to divisibility, or in regard to the ends. Now it is not possible to touch things infinite in regard to number in a finite time, but it is possible to touch things infinite in regard to divisibility: for time itself also is infinite in this sense. So that in fact we go through an infinite [space], in an infinite [time] and not in a finite [time], and we touch infinite things with infinite things, not with finite things." Philoponus, a sixth-century commentator (R.P. 136a, *Exc. Paris Philop. in Arist. Phys.* 803, 2. Vit.), gives the following illustration: "For if a thing were moved the space of a cubit in one hour, since in every space there are an infinite number of points, the thing moved must needs touch all the points of the space: it will then go through an infinite collection in a finite time, which is impossible."

is not another which you touch next: no two points are next each other, but between any two there are always an infinite number of others, which cannot be enumerated one by one.

(2) If, on the other hand, only the successive middle points are concerned, obtained by always halving what remains of the course, then the points are reached one by one, and, though they are infinite in number, they are in fact all reached in a finite time. His argument to the contrary may be supposed to appeal to the view that a finite time must consist of a finite number of instants, in which case what he says would be perfectly true on the assumption that the possibility of continued dichotomy is undeniable. If, on the other hand, we suppose the argument directed against the partisans of infinite divisibility, we must suppose it to proceed as follows:[11] "The points given by successive halving of the distances still to be traversed are infinite in number, and are reached in succession, each being reached a finite time later than its predecessor; but the sum of an infinite number of finite times must be infinite, and therefore the process will never be completed." It is very possible that this is historically the right interpretation, but in this form the argument is invalid. If half the course takes half a minute, and the next quarter takes a quarter of a minute, and so on, the whole course will take a minute. The apparent force of the argument, on this interpretation, lies solely in the mistaken supposition that there cannot be anything beyond the whole of an infinite series, which can be seen to be false by observing that 1 is beyond the whole of the infinite series $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$.

The second of Zeno's arguments is the one concerning Achilles and the tortoise, which has achieved more notoriety than the others. It is paraphrased by Burnet as follows:[12]

[11] Cf. Mr. C. D. Broad, "Note on Achilles and the Tortoise" [55], pp. 318–319.

[12] *Op. cit.*

"Achilles will never overtake the tortoise. He must first reach the place from which the tortoise started. By that time the tortoise will have got some way ahead. Achilles must then make up that, and again the tortoise will be ahead. He is always coming nearer, but he never makes up to it."[13]

This argument is essentially the same as the previous one. It shows that, if Achilles ever overtakes the tortoise, it must be after an infinite number of instants have elapsed since he started. This is in fact true; but the view that an infinite number of instants make up an infinitely long time is not true, and therefore the conclusion that Achilles will never overtake the tortoise does not follow.

The third argument,[14] that of the arrow, is very interesting. The text has been questioned. Burnet accepts the alterations of Zeller, and paraphrases thus:

"The arrow in flight is at rest. For, if everything is at rest when it occupies a space equal to itself, and what is in flight at any given moment always occupies a space equal to itself, it cannot move."

But according to Prantl, the literal translation of the unemended text of Aristotle's statement of the argument is as follows: "If everything, when it is behaving in a uniform manner, is continually either moving or at rest, but what is moving is always in the *now,* then the moving arrow is motionless." This form of the argument brings out its force more clearly than Burnet's paraphrase.

Here, if not in the first two arguments, the view that a finite part of time consists of a finite series of successive instants seems to be assumed; at any rate the plausibility of the argument seems to depend upon supposing that there are consecutive instants. Throughout an instant, it is said, a moving body is where it is: it cannot move during the instant, for that would require that the instant should have parts. Thus, suppose we consider a period consisting of a thousand instants, and suppose the arrow is in flight throughout this period. At each of the thousand instants, the arrow is where it is, though at the next instant it is somewhere else. It is never moving, but in some miraculous way the change of position has to occur *between* the instants, that is to say, not at any time whatever. This is what M. Bergson calls the cinematographic representation of reality. The more the difficulty is meditated, the more real it becomes. The solution lies in the theory of continuous series: we find it hard to avoid supposing that, when the arrow is in flight, there is a *next* position occupied at the *next* moment; but in fact there is no next position and no next moment, and when once this is imaginatively realised, the difficulty is seen to disappear.

[13] Aristotle's words are: "The second is the so-called Achilles. It consists in this, that the slower will never be overtaken in its course by the quickest, for the pursuer must always come first to the point from which the pursued has just departed, so that the slower must necessarily be always still more or less in advance." *Phys.* VI. 9, 239b (R.P. 138).

[14] *Phys.* VI. 9, 239b (R.P. 138).

The fourth and last of Zeno's arguments is[15] the argument of the stadium.

The argument as stated by Burnet is as follows:

First Position	Second Position
A	A . . . .
B . . . . →	B . . . .
C ←	C . . . .

"Half the time may be equal to double the time. Let us suppose three rows of bodies, one of which (A) is at rest while the other two (B, C) are moving with velocity in opposite directions. By the time they are all in the same part of the course, B will have passed twice as many of the bodies in C as in A. Therefore the time which it takes to pass C is twice as long as the time it takes to pass A. But the time which B and C take to reach the position of A is the same. Therefore double the time is equal to the half."

[15] *Phys.* VI. 9, 293b (R.P. 139).

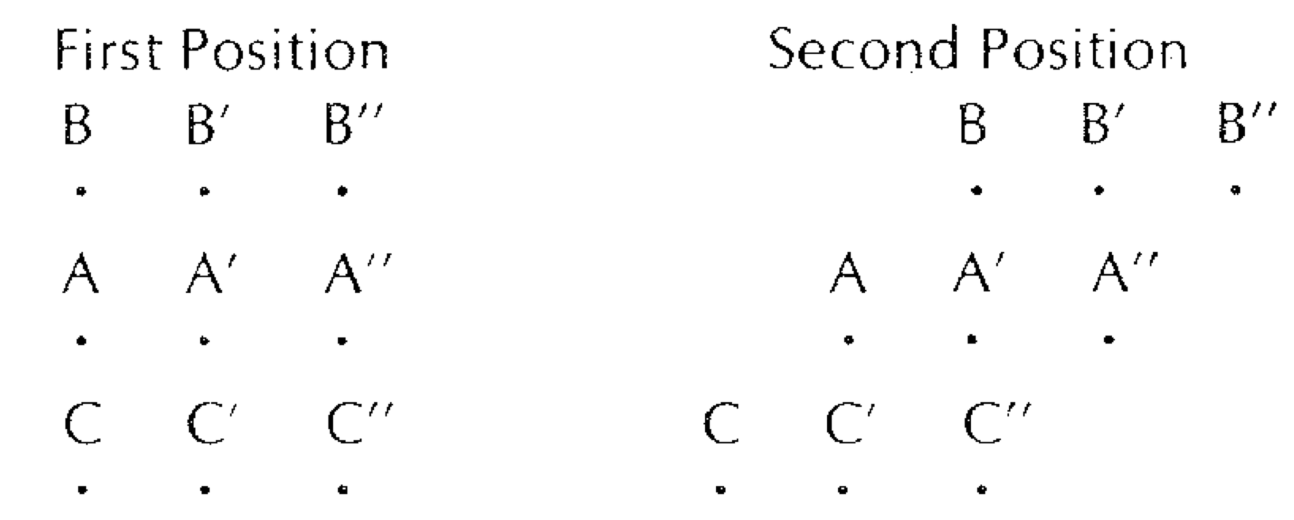
Gaye[16] devoted an interesting article to the interpretation of this argument. His translation of Aristotle's statement is as follows:

"The fourth argument is that concerning the two rows of bodies, each row being composed of an equal number of bodies of equal size, passing each other on a race-course as they proceed with equal velocity in opposite directions, the one row originally occupying the space between the goal and the middle point of the course, and the other that between the middle point and the starting-post. This, he thinks, involves the conclusion that half a given time is equal to double that time. The fallacy of the reasoning lies in the assumption that a body occupies an equal time in passing with equal velocity a body that is in motion and a body of equal size that is at rest, an assumption which is false. For instance (so runs the argument), let A A . . . be the stationary bodies of equal size, B B . . . the bodies, equal in number and in size to A A . . . , originally occupying the half of the course from the starting-post to the middle of the A's, and C C . . . those originally occupying the other half from the goal to the middle of the A's, equal in number, size, and velocity, to B B . . . . Then three consequences follow. First, as the B's and C's pass one another, the first B reaches the last C at the same moment at which the first C reaches the last B. Secondly, at this moment the first C has passed all the A's, whereas the first B has passed only half the A's and has consequently occupied only half the time occupied by the first C, since each of the two occupies an equal time in passing each A. Thirdly, at the same moment all the B's have passed all the C's: for the first C and the first B will simultaneously reach the opposite ends of the course, since (so says Zeno) the time occupied by the first C in passing each of the B's is equal to that occupied by it in passing each of the A's, because an equal time is occupied by both the first B and the

[16] *Loc. cit.* [25].

first C in passing all the A's. This is the argument: but it presupposes the aforesaid fallacious assumption."

This argument is not quite easy to follow, and it is only valid as against the assumption that a finite time consists of a finite number of instants. We may restate it in different language. Let us suppose three drill-sergeants, A, A′, and A″,



standing in a row, while the two files of soldiers march past them in opposite directions. At the first moment which we consider, the three men B, B′, B″ in one row, and the three men C, C′, C″ in the other row, are respectively opposite to A, A′, and A″. At the very next moment, each row has moved on, and now B and C″ are opposite A′. Thus B and C″ are opposite each other. When, then, did B pass C′? It must have been somewhere between the two moments which we supposed consecutive, and therefore the two moments cannot really have been consecutive. It follows that there must be other moments between any two given moments, and therefore that there must be an infinite number of moments in any given interval of time.

The above diffculty, that B must have passed C′ at some time between two consecutive moments, is a genuine one, but is not precisely the difficulty raised by Zeno. What Zeno professes to prove is that "half of a given time is equal to double that time." The most intelligible explanation of the argument known to me is that of Gaye.[17] Since, however, his explanation is not easy to set forth shortly, I will re-state

[17] *Op. cit.* [25], p. 105.

what seems to me to be the logical essence of Zeno's contention. If we suppose that time consists of a series of consecutive instants, and that motion consists in passing through a series of consecutive points, then the fastest possible motion is one which, at each instant, is at a point consecutive to that at which it was at the previous instant. Any slower motion must be one which has intervals of rest interspersed, and any faster motion must wholly omit some points. All this is evident from the fact that we cannot have more than one event for each instant. But now, in the case of our A's and B's and C's, B is opposite a fresh A every instant, and therefore the number of A's passed gives the number of instants since the beginning of the motion. But during the motion B has passed twice as many C's, and yet cannot have passed more than one each instant. Hence the number of instants since the motion began is twice the number of A's passed, though we previously found it was equal to this number. From this result, Zeno's conclusion follows.

Zeno's arguments, in some form, have afforded grounds for almost all the theories of space and time and infinity which have been constructed from his day to our own. We have seen that all his arguments are valid (with certain reasonable hypotheses) on the assumption that finite spaces and times consist of a finite number of points and instants, and that the third and fourth almost certainly in fact proceeded on this assumption, while the first and second, which were perhaps intended to refute the opposite assumption, were in that case fallacious. We may therefore escape from his paradoxes either by maintaining that, though space and time do consist of points and instants, the number of them in any finite interval is infinite; or by denying that space and time consist of points and instants at all; or lastly, by denying the reality of space and time altogether. It would seem that Zeno himself, as a supporter of Parmenides, drew the last of these three possible deductions, at any rate in regard to time. In this a very large number of philosophers have followed him. Many others, like M. Bergson, have preferred to deny that space and time consist of points and instants. Either of these solutions will meet the difficulties in the form in which Zeno raised them. But, as we saw, the difficulties can also be met if infinite numbers are admissible. And on grounds which are independent of space and time, infinite numbers, and series in which no two terms are consecutive, must in any case be admitted. Consider, for example, all the fractions less than 1, arranged in order of magnitude. Between any two of them, there are others, for example, the arithmetical mean of the two. Thus no two fractions are consecutive, and the total number of them is infinite. It will be found that much of what Zeno says as regards the series of points on a line can be equally well applied to the series of fractions. And we cannot deny that there are fractions, so that two of the above ways of escape are closed to us. It follows that, if we are to solve the whole class of difficulties derivable from Zeno's by analogy, we must discover some tenable theory of infinite numbers. What, then, are the difficulties which, until the last thirty years, led philosophers to the belief that infinite numbers are impossible?

The difficulties of infinity are of two kinds, of which the first may be called sham, while the others involve, for their solution, a certain amount of new and not altogether easy thinking. The sham difficulties are those suggested by the etymology, and those suggested by confusion of the mathematical infinite with what philosophers impertinently call the "true" infinite. Etymologically, "infinite" should mean "having no end." But in fact some infinite series have ends, some have not; while some collections are infinite without being serial, and can therefore not properly be regarded as either endless or having ends. The series of instants from any earlier one to any later one (both included) is infinite, but has two ends; the series of instants from the beginning of time to the present moment has one end, but is infinite. Kant, in his first antinomy, seems to hold that it is harder for the past to be infinite than for the future to be so, on the ground that the past is now completed, and that nothing infinite can be completed. It is very difficult to see how he can have imagined that there was any sense in this remark; but it seems most probable that he was thinking of the infinite as the "unended." It is odd that he did not see that the future too has one end at the present, and is precisely on a level with the past. His regarding the two as different in this respect illustrates just that kind of slavery to time which, as we agreed* in speaking of Parmenides, the true philosopher must learn to leave behind him.

The confusions introduced into the notions of philosophers by the so-called "true" infinite are curious. They see that this notion is not the same as the mathematical infinite, but they choose to believe that it is the notion which the mathematicians are vainly trying to reach. They therefore inform the mathematicians, kindly but firmly, that they are mistaken in adhering to the "false" infinite, since plainly the "true" infinite is something quite different. The reply to this is that what they call the "true" infinite is a notion totally irrelevant to the problem of the mathematical infinite, to which it has only a fanciful and verbal analogy. So remote is it that I do not propose to confuse the issue by even mentioning what the "true" infinite is. It is the "false" infinite that concerns us, and we have to show that the epithet "false" is undeserved.

There are, however, certain genuine difficulties in understanding the infinite, certain habits of mind derived from the consideration of finite numbers, and easily extended to infinite numbers under the mistaken notion that they represent logical necessities. For example, every number to which we are accustomed, except 0, has another number immediately before it, from which it results by adding 1; but the first infinite number does not have this property. The numbers before it form an infinite series, containing all the ordinary finite numbers, having no maximum, no last finite number, after which one little step would plunge us

*[Russell, *Our Knowledge of the External World* [100], p. 181.]

into the infinite. If it is assumed that the first infinite number is reached by a succession of small steps, it is easy to show that it is self-contradictory. The first infinite number is, in fact, beyond the whole unending series of finite numbers. "But," it will be said, "there cannot be anything beyond the whole of an unending series." This, we may point out, is the very principle upon which Zeno relies in the arguments of the race-course and the Achilles. Take the race-course: there is the moment when the runner still has half his distance to run, then the moment when he still has a quarter, then when he still has an eighth, and so on in a strictly unending series. Beyond the whole of this series is the moment when he reaches the goal. Thus there certainly can be something beyond the whole of an unending series. But it remains to show that this fact is only what might have been expected.

The difficulty, like most of the vaguer difficulties besetting the mathematical infinite, is derived, I think, from the more or less unconscious operation of the idea of *counting*. If you set to work to count the terms in an infinite collection, you will never have completed your task. Thus, in the case of the runner, if half, three-quarters, seven-eighths, and so on of the course were marked, and the runner was not allowed to pass any of the marks until the umpire said "Now," then Zeno's conclusion would be true in practice, and he would never reach the goal.

But it is not essential to the existence of a collection, or even to knowledge and reasoning concerning it, that we should be able to pass its terms in review one by one. This may be seen in the case of finite collections; we can speak of "mankind" or "the human race," though many of the individuals in this collection are not personally known to us. We can do this because we know of various characteristics which every individual has if he belongs to the collection, and not if he does not. And exactly the same happens in the case of infinite collections: they may be known by their characteristics although their terms cannot be enumerated. In this sense, an unending series may nevertheless form a whole, and there may be new terms beyond the whole of it.

Some purely arithmetical peculiarities of infinite numbers have also caused perplexity. For instance, an infinite number is not increased by adding one to it, or by doubling it. Such peculiarities have seemed to many to contradict logic, but in fact they only contradict confirmed mental habits. The whole difficulty of the subject lies in the necessity of thinking in an unfamiliar way, and in realising that many properties which we have thought inherent in number are in fact peculiar to finite numbers. If this is remembered, the positive theory of infinity . . . will not be found so difficult as it is to those who cling obstinately to the prejudices instilled by the arithmetic which is learnt in childhood.

HENRI BERGSON

The Cinemato-graphic View of Becoming

Now, if we try to characterize more precisely our natural attitude towards Becoming, this is what we find. Becoming is infinitely varied. That which goes from yellow to green is not like that which goes from green to blue: they are different *qualitative* movements. That which goes from flower to fruit is not like that which goes from larva to nymph and from nymph to perfect insect: they are different *evolutionary* movements. The action of eating or of drinking is not like the action of fighting: they are different *extensive* movements. And these three kinds of movement themselves—qualitative, evolutionary, extensive—differ profoundly. The trick of our perception, like that of our intelligence, like that of our language, consists in extracting from these profoundly different becomings the single representation of becoming *in general,* undefined becoming, a mere abstraction which by itself says nothing and of which, indeed, it is very rarely that we think. To this idea, always the same, and always obscure or unconscious, we then join, in each particular case, one or several clear images that represent *states* and which serve to distinguish all becomings

From Henri Bergson, *Creative Evolution,* trans. Arthur Mitchell (New York: Holt, Rinehart and Winston, 1911), chap. 4, pp. 304–311.