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Incommensurables and Incomparables: On the Conceptual Status and the Philosophical Use of Hyperreal Numbers

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Abstract After briefly considering the ancient Greek and nineteenth-century history of incommensurables (magnitudes that do not have a common aliquot part) and incomparables (magnitudes such that the larger can never be surpassed by any finite number of additions of the smaller to itself), this paper undertakes two tasks. The first task is to consider whether the numerical accommodation of incommensurables by means of the extension of the ordered field of rational numbers to the field of reals is 'similar' or analogous to the numerical accommodation of incomparables by means of the extension of the ordered field of reals to the field of hyperreals. The second task is to evaluate several contemporary attempts to use concepts and techniques of the nonstandard mathematics of hyperreals to address classical, Zenonian puzzles concerning continuous magnitudes. The result of both these undertakings is, in a certain sense, 'deflationary'.

1 Introduction We are now quite accustomed to the view that the extension of the field of rational numbers to the field of real numbers is a natural, useful, and proper move—irrespective of the precise mathematical methods that we may prefer in order to effect that extension and irrespective of our ontological persuasions (or lack thereof) with respect to (rational and real) numbers and other mathematical objects. To someone of a philosophical cast of mind contemplating the development of nonstandard analysis over the last forty years, it might well be assumed that, by parity of whatever reasoning that could be invoked to justify this view concerning the relation of the rationals to the reals, we should similarly be prepared to regard the extension of the field of real numbers to the field of hyperreal numbers as a natural, useful, and proper move.

I begin by drawing a distinction: *incommensurables* are magnitudes that do not have a common aliquot part; what I call *incomparables* are magnitudes such that the

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larger can never be surpassed by any finite number of additions of the smaller to itself. After briefly rehearsing some well-known facts about the ancient Greek and nineteenth-century history of these concepts, I shall make a case for the initial appeal of seeing the extension of the reals to the hyperreals as 'similar' to the extension of the rationals to the reals but then raise some animadversions against this perspective. I conclude that the vagueness or ambiguity of the (informal) notion of 'similarity' entails that there is no obviously correct answer to the question of whether the extension of the reals to the hyperreals is similar to the extension of the rationals to the reals. I shall then argue that the contemporary development, by so-called 'nonstandard analysis', of the mathematics of hyperreal numbers has what I term a 'Janus-faced' character (which I attempt to describe more precisely) that militates against a paradigm-shift that would result in the displacement of the real by the hyperreal number line. This Janus-faced character proves to be particularly relevant with respect to contemporary attempts to develop novel, nonstandard accounts of some classical puzzles concerning continuous magnitudes, such as those of Zeno of Elea. I conclude that these attempts encounter a problem that arises as a result of this Janus-faced character of the principal contemporary developments of nonstandard mathematics.

2 A brief historical background According to canonical lore, it was those mythical, mystical Pythagoreans who discovered that there are some magnitudes that are incommensurable with others. In addition to constituting a specifically mathematical conundrum, this discovery supposedly was regarded as generally very bad news by the Pythagoreans who wished to develop a numerical ontology of everything. A common assumption is that the first instance of incommensurability to be so unpropitiously discovered was that of the diagonal and side of a square. Aristotle alludes to a *reductio* in which it is proved that "the diagonal is incommensurable [with the side] because what is odd becomes equal to what is even if it is assumed to be commensurable" (*An. pr.* 1.23.41a26–27). Such a proof is easy enough to construct: a version of it that was formerly printed as Proposition 117 of the tenth book of Euclid is now thought to be an interpolated scholium. In contemporary (and quite un-Greek) terminology, we see that the rational numbers are not closed under the operation of taking the square root: $\sqrt{2}$ (length of the diagonal of a square with unit side) is not expressible as a ratio of integers. In this sense—a sense that obviously has implications even for elementary plane geometry— the rational numbers are 'incomplete'. Would it not make good sense to 'fill in the gaps' by adding to the rationals some 'new' numbers by which we can represent the length of the diagonal of the unit square, as well as other constant, incommensurable (and, in fact, 'transcendental') ratios such as that of the circumference of a circle to its diameter.

Sensible or not, it was a move that Greek geometers did not make. Rather, the Greek response was to draw a sharp distinction between discrete πλῆθος (multiplicity or plurality) and continuous μέγαθος (magnitude) and to locate the phenomenon of incommensurability squarely in the latter category. Distinguishing these two coordinate kinds of quantity πόσον, Aristotle characterizes multiplicity as numerable quantity and magnitude as measurable quantity (*Meta.* 5.13.1020a8-10), maintaining that "number (ἀριθμός) is commensurable (σύμμετρος), but number is not predicated with respect to what is not commensurable" (*Meta.* 5.15.1021a5). The 'theory of incommensurability' is explicated by Euclid in his theory of proportionality *of magnitudes* in the fifth and tenth books of the *Elements*. Bostock nicely characterizes the Greek persistence in refusing to subsume (relatively) incommensurable magnitudes under the concept of number. As he points out, the assumption "that every line must have a definite length could perfectly well be denied—classical Greek mathematics did in fact deny it—but it is surely this sort of assumption that lies behind our conviction that there is such a number as $\sqrt{2}$" (Bostock [2], pp. 210–11).

A second Greek mathematical conundrum pertains to what I have termed incomparable magnitudes. In his work, the *Methodus* (῞Εφοδος), the great Hellenistic mathematician Archimedes of Syracuse (third century B.C.) sets forth a method of discovery that he used in finding some of his most famous results. Examples of what is often termed the Archimedean 'mechanical method' are set forth in this work, which was known in antiquity but lost thereafter and rediscovered only in 1899. In brief, the method involves an 'idealization' of the concept of center of gravity that applies to straight lines and plane figures as well as solids (in *De planorum aequilibriis sive de centris gravitatis planorum*). For example, the center of gravity of a straight line is its midpoint; the center of gravity of any parallelogram is the point of intersection of its diagonals; the center of gravity of a triangle is the point at which the straight lines joining each vertex to the midpoint of the opposite side meet. It also involves the assumption that a plane figure can be identified with the collection of (parallel) linear segments 'filling it up' and a solid figure can be identified with the collection of its (parallel) planar laminae or 'cross sections'. The strategy is to 'balance' the linear segments or planar laminae of one geometrical figure against those of those of another and thus to infer the ratio in which (the areas or volumes of) the complete figure stand.

It is important to emphasize that Archimedes did not consider this sort of argument to constitute a *proof* of the result. Early and more-or-less rigorous forms of what is now often termed the Zenonian paradox of measure were well known by the third century B.C.: on pain of paradox, the positive breadth (πλάτος) of a plane figure cannot be identified with the sum of the breadths of linear sections of the figure which individually have null breadth; nor can the positive depth (βάθος) of a solid figure be identified with the sum of the depths of planar cross sections which individually have null depth.

Would it not make good sense to introduce the concept of positive but 'vanishingly small' or 'infinitesimal' magnitudes so that, say, the linear segments would not be strictly *lines*—they would have positive breadth—but such extremely small breadth that their centers of gravity would be determined exclusively by their length (μῆκος), just as is the case for lines in the strict sense. Similarly, planar laminae of a solid figure would not be strictly *planes*—they would have positive depth—but such extremely small depth that their centers of gravity would be determined exclusively by their two-dimensional areas. Such a small breadth or depth would be 'vanishingly small', 'insignificant', or 'unassignable' relative to any 'standard' magnitude in the following sense. It would not 'have a ratio' (λόγον ἔχειν) to the larger standard magnitude in the terms of Definition 4 of the fifth book of the *Elements*: "a magnitude is said to have a ratio to another when it can, when multiplied, exceed the other." As previously indicated, I use the term 'incomparable' to designate such magnitudes that would not (as we now say) stand in any real-valued ratio to one another.[1]

Again, whether sensible or not, the introduction of quantities that are incomparable in this sense seems not to have been a move made within the mainstream of Greek mathematics. In fact, Archimedes is known for having stated in the first book of *De sphaera et cylindro* a postulate (the Archimedean or Eudoxean axiom) that requires that the difference between any two unequal magnitudes of the same kind (i.e., lines, surfaces, solids) and any other magnitude of that kind 'have a ratio to one another' in the sense of Definition 4 of the fifth book of Euclid. For present purposes, I adopt the familiar forms the Archimedean axiom: (1) for $x > y > 0$, there is some positive natural number n such that y added to itself n times exceeds x; or (2) (a common contemporary account that is equivalent, for ordered fields, to the first account) for each element x of the field, there is a positive number n such that the multiplicative identity element (usually designated '1') added to itself n times exceeds x.

So, adoption of the Archimedean axiom rules out the possibility of incomparable magnitudes: it rules out magnitudes that would be infinitesimal relative to standard, finite quantities as well as infinitely large quantities, which would be larger than any standard finite quantities and equal to the multiplicative inverses of infinitesimals. The combination, then, of the Archimedean axiom and *reductio* proof provides a means of utilizing an informal concept limit in a finitistic context (*viz.*, the availability of *finite* differences between variable quantities V and some fixed quantity L, that can be made 'as small as one wishes') without actually 'passing to the limit'—that is, without, for example, actually identifying the area of the parabolic segment with the 'infinite sum' of areas of inscribed triangles. This perspective became the mathematically orthodox one, the unfortunately named 'method of exhaustion'.

In concluding this brief account of incommensurables and incomparables in Greek mathematics, I wish to emphasize the point that the Greeks made no attempt to accommodate either incommensurables or incomparables *numerically*. However, classical Greek mathematics found comfortable (if, at times, a bit complicated and unwieldy) geometrical means for dealing with both phenomena: the theory of proportion for the analysis of incommensurables; a combination of the method of exhaustion, as a method of proof, and (at least in some cases) the method of indivisibles, as a method of discovery, for infinitary analysis.

According to contemporary mathematical orthodoxy, a satisfactory and final resolution of both the ancient conundrums that we have been considering—that pertaining to incomparables (and, more generally, the employment of infinitary analytical techniques) and that pertaining to incommensurables—was achieved in the nineteenth century. Although he was in certain respects anticipated by Cauchy and Bolzano, Weierstrass is usually credited with fully appreciating that the limit concept could be developed, in terms of the so-called ε, δ approach, in such a way as to eliminate any temptation to appeal to an idea of infinitesimal or infinitely small (and infinitely great) quantities. Talk of 'infinitesimal differences' or differentials increasingly came to be regarded as a mere *façon d'parler* (see Robinson [25], ch. 10). Despite the still prevalent Leibnizian notation, $(df/dx)_{x=x_0}$, the derivative is not some ratio of infinitesimal 'differences' or quantities.[2] Indeed, the Weierstrassian perspective became the mathematically orthodox one: incomparables were denied any *numerical* or *arithmetic* status.

The nineteenth-century fate of incommensurables was, in a certain respect, diametrically opposed to the nineteenth-century fate of incomparables: with the arithmeticization of the continuum by Dedekind and Cantor, relatively incommensurable magnitudes found a secure *numerical* home as irrationals. Dedekind and Cantor each developed techniques, now regarded as standard, for the 'construction' of the real numbers from the rationals: reals are defined in terms of disjoint nonempty classes ('cuts') of rationals by Dedekind and in terms of Cauchy sequences by Cantor.[3]

The principal issue that motivated both Dedekind and Cantor was the construction of an *arithmetic continuum* as an (analytic or reductive) instrument for representing the *geometrical continuum.* Their aim, in other words, was the arithmeticization of the continuous magnitudes ($\mu\varepsilon\gamma\varepsilon\theta\hat{\eta}$) that the Greeks have relegated to geometry (the study of continuous magnitudes), as opposed to arithmetic (the study of discrete magnitudes). Ehrlich aptly characterizes this program:

> The newly constructed ordered field of real numbers was dubbed the *arithmetic continuum* because it was held that this number system is completely adequate for the analytic representation of all types of continuous phenomena. In accordance with this view, the *geometric linear continuum* was assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to insure this would be the case. In honor of Cantor and Dedekind, who first proposed the thesis, the presumed correspondence between the two structures has come to be called *the Cantor-Dedekind axiom.* Given the Archimedean nature of the real number system, once this axiom is adopted we have the classic result that infinitesimal line segments are superfluous to the analysis of the structure of a continuous straight line (Ehrlich [7], p. viii).

In other words, as a result of nineteenth-century developments with respect to the two classical Greek phenomena of incommensurable and incomparable magnitudes, the former was arithmeticized—incorporated into a system of numbers—while the latter was not.

The Cantor-Dedekind perspective is perhaps given added support by a classical uniqueness result for the arithmetic continuum or ordered field of real numbers:

> (Real Completeness Theorem): The ordered field $\mathbb{R}$ of real numbers is, up to order-isomorphism, the one ordered field that is *complete* in the following sense: an ordered field $\mathbb{F}$ is complete just in case every nonempty subset $X \subseteq \mathbb{F}$ that has an upper bound in $\mathbb{F}$ has a least upper bound (lub or supremum) in $\mathbb{F}$.[4]

It is also well known that the completeness of an ordered field is equivalent to several other important properties, for example, (A) the conjunction of the field's possessing the Archimedean property and the existence, for every Cauchy sequence definable on the field, of an element of the field that is the limit of that sequence; (B) Dedekind continuity (the absence of gaps—that is, the absence of any cut (X, Y), of the ordered field $\mathbb{F}$, where both X and Y are nonempty, with respect to which there is in $\mathbb{F}$ *both* no greatest member of X and no least member of Y). This sense of 'completeness' should be distinguished from another sense in which the ordered field $\mathbb{R}$ is also uniquely complete. $\mathbb{R}$ is, up to isomorphism, the unique Archimedean ordered field that is *Archimedean complete* in the following sense: there is no proper extension of it that is itself an Archimedean ordered field. Ehrlich has recently discussed the way in which Hahn obtained this result as a special case in the course of his study of Archimedean complete ordered *groups*, which may be non-Archimedean (Ehrlich [8]).[5]

It might thus seem that any proper extension of the field $\mathbb{R}$ of real numbers must sacrifice in some way the 'completeness' of the reals. And, in a sense, this is true: any such extension must sacrifice the Archimedean property of the ordered field. But we already knew that the addition of incomparable magnitudes (infinitesimals and infinitely large numbers, relative to the standard reals) to the ordered field of reals would entail sacrifice of the Archimedean property. The question is whether there are *other* significant senses of 'completion' in which an ordered field containing representations of incomparable magnitudes might be regarded as a completion of $\mathbb{R}$.

3 *The hyperreals: A non-Archimedean extension of the reals* The model-theoretic construction of the hyperreals and an accompanying theory of nonstandard analysis developed by Robinson in the 1960s has become well known, and Robinson is sometimes credited with restoring what I am calling incomparables—in particular, infinitesimals—to mathematical grace as *numerical* entities. However, as Ehrlich has pointed out, "Robinson was an authority on the theory of ordered algebraic systems before he became a nonstandard analyst" ([7], p. xxii) and, as such, Robinson may be regarded as extending earlier work on non-Archimedean ordered fields. Ehrlich [7] and [8] provide a fine survey of a tradition of work on non-Archimedean geometry and algebra extending from about the 1870s into the mid-twentieth century that includes such figures as Thomae, du Bois-Reymond, Stolz, Veronese, Vivanti, Bettazzi, Hilbert, Hölder, Hahn, and (later) Artin, Schreier, Tarski, and McKinsey. Although much of this work no doubt should be characterized, in Ehrlich's words, as "mainstream mathematics," there is a sense in which it represents a minority tradition. It is arguable that the Cantor-Dedekind perspective eventually achieved the status of mathematical orthodoxy despite continuing work in non-Archimedean geometry and algebra. The attitude of Cantor himself seems to have played a part in this process. His own opposition to infinitesimals was unswerving and, at times, vitriolic. In a letter to Vivanti, he credited Thomae with being the first to "infect mathematics with the Cholera-Bacillus of infinitesimals;" and he suggests that, in developing the ideas of Thomae, du Bois-Reymond found "excellent nourishment for the satisfaction of his own burning ambition and conceit." (Letter of Cantor to Vivanti, quoted in Dauben [5], p. 131). In effect, Cantor denied the possibility of infinitesimal numbers because he believed that the Archimedean principle, or something equivalent to it, was entailed by the concept of (linear) number. In the words of Dauben,

> had Cantor agreed that the Archimedean property of the real numbers was merely axiomatic, then there was no reason to prevent the development of number systems by merely denying the axiom, so long as consistency was still preserved. But to have allowed this would have left Cantor open to the challenge that, if infinitesimals could be produced without contradiction, then his own view of the continuum was lacking and the completeness of his own theory of number would have been contravened. ([5], p. 235)

Of course, as has been previously noted, no non-Archimedean ordered field is complete in the mathematical sense specified above—or in any intuitive sense that is equivalent to or entailed by the Archimedean axiom (e.g., in having no gaps). However, an ordered field is characterized not just by the order relation but by the algebraic field operations; and if we shift our attention from the order relation to these algebraic operations, a rather different intuitive picture of 'completeness' emerges.

> An ordered field $\mathbb{F}$ is *real closed* just in case both (i) every positive element of $\mathbb{F}$ has a square root in $\mathbb{F}$ and (ii) every polynomial of odd degree has a root in $\mathbb{F}$.

Artin and Schreier investigated the theory of real closed fields in the 1920s showing that the ordered field $\mathbb{R}$ of real numbers is a real closed field. The ordered field $\mathbb{Q}$ of rationals clearly is not: there is no element of $\mathbb{Q}$, for example, that is the square root of the element 2. There is a fairly intuitive sense in which the concept of a real closed field represents a sort of 'algebraic completeness'. In fact, Artin and Schreier's work formed the basis of the discipline of 'real algebra'. Sinaceur points out that Artin and Schreier termed the propositions provable about the real numbers within the theory of real closed fields the " 'theorems of real algebra', thus creating a new discipline which screens the usual propositions of real analysis in order to locate those which can be assigned or reassigned to algebra. In general, every question involving real numbers which is solvable within the general framework of real closed fields is an algebraic question" (Sinaceur [27], p. 196). However, $\mathbb{R}$ is *not* characterizable up to isomorphism as the unique real closed field. There are real closed fields of only denumerable cardinality, such as the field $\mathbb{R}_{\mathbb{A}}$ of algebraic numbers; and there are extensions of the reals, such as Robinson's ordered field $^*\mathbb{R}$ of hyperreals (to be further discussed) and Conway's ordered field of 'surreal' numbers, that are real closed fields (see Conway [4] and Ehrlich [6]).

In the late 1940s, Tarski and McKinsey showed that all real closed fields, considered as models of first-order sentences of the language of ordered fields, are elementarily equivalent—that is, satisfy the same set of first-order sentences of the language of ordered fields. More particularly, for the real closed field $\mathbb{R}$ of reals, if $\mathbb{F}$ is a real closed field and $\mathbb{R} \subseteq \mathbb{F}$, then $\mathbb{F}$ is an elementary extension of $\mathbb{R}$—that is, any first-order formula of the language of ordered fields containing only constants that designate real numbers is true when interpreted in $\mathbb{R}$ if and only it is true when interpreted in $\mathbb{F}$. Keisler points out the limitation of this classical and very important result:

> There is an important loophole in the result of Tarski. It applies only to formulas in the language of ordered fields, that is, formulas built up from predicates $=, \leq$ the function symbols $+, \cdot$, and the constants 0, 1. Nothing has been said about how one might extend other real relations or functions to the larger set $\mathbb{F}$. For example, the exponential functions, the trigonometric functions, and the set of natural numbers cannot be defined in the language of ordered fields (Keisler [17], p. 213).

Keisler proceeds to note that a construction, such as Robinson's, of the (non-Archimedean) ordered field $^*\mathbb{R}$ of hyperreal numbers will yield the result that $^*\mathbb{R}$ "is an elementary extension of the real number system even in the full language which has a symbol for every relation and function over $\mathbb{R}$" ([17], p. 213).

The existence of non-Archimedean elementary extensions of the reals, such as $^*\mathbb{R}$, can be established using the compactness result for first-order logic.[6] Contemporary nonstandard analysis supplies a number of ways of proceeding to introduce the non-Archimedean ordered field of hyperreals, as Keisler notes: "either by an axiomatic approach which lists its properties and proves that a structure with those properties exists, or by an explicit construction" ([17], p. 213). With a slightly different emphasis, Chang and Keisler comment that "in practice most research in the subject [of what they call 'Robinsonian analysis'] uses one of two approaches: superstructures and internal set theory. The intuitive idea underlying all approaches is to start with a set-theoretic universe $\langle V, \in\rangle$ and form an elementary extension $\langle W, E\rangle$ in which all infinite sets are enlarged" (Chang and Keisler [3], p. 263). The "explicit construction" to which Keisler refers is a certain kind of ultrapower, which establishes the existence of superstructures possessing the desired properties. So we can classify the most frequently encountered approaches to nonstandard structures into the model-theoretic (superstructures—often developed with the use of ultrapowers) and set-theoretic (usually some extension/modification of Zermelo-Fraenkel set theory with the axiom of choice [ZFC]). I shall describe the former approach and, later, make some brief comments about the latter.

3.1 Superstructures The superstructure approach works with languages with a constant designating the binary set-membership relation $\in$ and bounded quantifiers defined in terms of it [$(\forall x \in y)$, $(\exists x \in y)$]. We shall let the base set R of a superstructure $V(\mathrm{R})$ be the set of real numbers. The nth cumulative power set is defined inductively as $V_0(\mathrm{R}) = \mathrm{R}$; $V_{n+1}(\mathrm{R}) = V_n(\mathrm{R}) \cup \mathcal{P}(V_n(\mathrm{R}))$. Then the superstructure $V(\mathrm{R})$ over R is the union of all the cumulative power sets V_i. In view of the standard method for defining ordered sequences in terms of sets, n-ary functions as $n+1$-ary relations, and so on, it is fairly clear that a superstructure contains the properties of reals, relations on reals, functions and operations on reals, "function spaces, measures, and all other structures from classical analysis" (Keisler [16], pp. 39–40). The 'trick' of the method of superstructures is to find *another* superstructure $V(^*\mathrm{R})$ and a *monomorphism* or injective mapping * from $V(\mathrm{R})$ into $V(^*\mathrm{R})$ satisfying certain conditions.[7] The result is an ordered triple $\langle V(\mathrm{R}), V(^*\mathrm{R}), *\rangle$ that Chang and Keisler call a *nonstandard universe* ([3], pp. 266–7). Let $\langle V(\mathrm{R}), \in\rangle$ be a model with $\in$ the binary set-membership relation. The essential conditions that $\langle V(\mathrm{R}), V(^*\mathrm{R}), *\rangle$ must satisfy in order to be a nonstandard universe are (in addition to the requirement that R and *R be infinite, which we have already assumed):

(i) that (as the notation already suggests) * maps the base set R of $V(\mathrm{R})$ to the base set *R of $V(^*\mathrm{R})$;

(ii) that the mapping * be a bounded elementary embedding of $\langle V(\mathrm{R}), \in\rangle$ into $\langle V(^*\mathrm{R}), \in\rangle$—that is, for every bounded-quantifier well-formed formula φ with n free variables (where φ is a bounded-quantifier wff *of classical analysis*) and every $a_1, \ldots, a_n \in V(\mathrm{R})$, $a_1, \ldots, a_n$ satisfy φ in $\langle V(\mathrm{R}), \in\rangle$ (i.e., $\langle V(\mathrm{R}), \in\rangle \models \varphi[a_1, \ldots, a_n]$) if and only if $^*a_1, \ldots, ^*a_n$ satisfy φ in $\langle V(^*\mathrm{R}), \in\rangle$ (i.e., $\langle V(^*\mathrm{R}), \in\rangle \models \varphi[^*a_1, \ldots, ^*a_n]$).[8].

(iii) that, for every infinite $X \subseteq R$, ${}^\sigma X = \{^*r : r \in X\} \subsetneq X$.

We also desire that the nonstandard universe $\langle V(\mathrm{R}), V(^*\mathrm{R}), *\rangle$ be *saturated over* $V(\mathrm{R})$. There are a number of ways of characterizing saturation in such a case. One way is the following. Suppose that we extend our bounded-quantifier language for the reals (with a constant '$\in$' for set membership) by adding a constant for each element of $V(\mathrm{R})$. Then

> $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ is saturated over $V(\mathrm{R})$ if and only if, for every $n < \omega$ and for every set Γ of bounded-quantifier formulas of this expanded language, *if* every *finite* $\Delta \subseteq \Gamma$ is satisfiable in $\langle V(\mathrm{R}), \in\rangle$ by elements of $V_n(\mathrm{R})$, then Γ is satisfiable in $\langle V(^*\mathrm{R}), \in\rangle$ by an element c of $V_n(^*\mathrm{R})$. ([3], p. 283)

Although there are other accounts of saturation,[9] the preceding account nicely suggests how nonstandard universes introduce *numerical* elements (members of non-Archimedean ordered fields) to represent quantities that were once regarded as 'ideal' in something like the sense of Leibniz: mathematical fictions introduced to shorten the process of reasoning. For example, saturation yields the result that, since there is some $N > n$ for each n that is a member of any *finite* subset of the set $\mathbb{N}$ of natural numbers, there is some 'ideal' element $c \in {}^*\mathbb{N}$ such that $c \mathbin{^*>} {}^*n$, for every $n \in \mathbb{N}$. It can then be shown that there is another 'ideal' numerical element $c^{-1} = 1/c \in V(^*\mathrm{R})$, the multiplicative inverse of c, that is infinitesimal, nonzero, and nonnegative but less than ($^*<$) any standard hyperreal ('embedded real') $^*r$ such that $r \in \mathrm{R}$.

Along with the saturation characteristic of nonstandard universes, the other especially crucial characteristic of nonstandard universes is that the monomorphism * of such universes be characterized by what is usually referred to as the *transfer principle* (or *Leibniz's principle*). In essence, this is the requirement (ii) above: the requirement that the monomorphism * be a bounded elementary embedding of $V(\mathrm{R})$ into $V(^*\mathrm{R})$. We thus obtain a way of taking any sentence 'of the language of the reals' and its intended interpretation in $V(\mathrm{R})$ and constructing, with the use of the monomorphism *, a hyperreal interpretation of it (its '*-transform') in $V(^*\mathrm{R})$. There is a shift from 'semantics' to 'syntax' here that, while natural enough from the mathematical perspective, can be confusing to philosophers whose mathematical training is primarily logical. As previously noted (see note 8), condition (ii) in the definition of a nonstandard universe is semantical: it pertains to the equivalent satisfaction of the *same* bounded-quantifier formulas (of the theory of real analysis) by two different models, related by the *semantic* monomorphism *. However, in the development and use of nonstandard analysis it has become customary to turn the semantic monomorphism * into a *syntactic* transformation. That is, * in its syntactic sense is an inductively defined mapping of terms or canonical names for entities in the domain $V(\mathrm{R})$ of the model $\langle V(\mathrm{R}), \in\rangle$ onto terms or canonical names for entities in the domain $V(^*\mathrm{R})$ of the model $\langle V(^*\mathrm{R}), \in\rangle$—entities that are the *-images of the denotations of the corresponding terms of the language of real analysis interpreted by $\langle V(\mathrm{R}), \in\rangle$. While generally harmless enough, and indeed often quite useful from a mathematical perspective, this ambiguity of the monomorphism * results in a subtle shift in the conception of a nonstandard universe. We began, it seems, with the conception of *one* language and *one* 'theory' (e.g., that of real analysis or some proper subtheory of it) with two models—one standard and one nonstandard. When the monomorphism * is rendered syntactic, however, we seem to have created *two* languages and *two* theories. There is the language/theory of real analysis, to be interpreted in the standard model $\langle V(\mathrm{R}), \in\rangle$ of the reals; and there is a *certain part* of the language/theory of (nonstandard) hyperreal analysis—the part that 'corresponds', in terms of the monomorphism *, to the language/theory of the reals—which is to be interpreted in the nonstandard

model $\langle V(^*\mathrm{R}), \in\rangle$. The latter, syntactic perspective gives rise to questions concerning the status of that part of the 'language/theory of hyperreals' (i.e., the theory of the model $\langle V(^*\mathrm{R}), \in\rangle$) that does *not* 'correspond' to the theory of the reals. It is here that what I termed the 'Janus-faced character' of contemporary nonstandard mathematics manifests itself—an issue to which I shall later return.

However, it is the syntactic use of * that gives rise to what is usually called the transfer principle or Leibniz's principle:

> a bounded well-formed formula (wff) φ is true (in $\langle V(\mathrm{R}), \in\rangle$) if and only if its *- transform $^*\varphi$ (an inductively defined syntactic transformation of φ in which its terms are replaced by terms canonically designating the *-images in $\langle V(^*\mathrm{R}), \in\rangle$ of the denotations of those original terms in $\langle V(\mathrm{R}), \in\rangle$) is true (in $\langle V(^*\mathrm{R}), \in\rangle$).

Consequently, a 'nonstandard theory of the hyperreals' can be regarded as a *conservative extension* of the '(standard, Archimedean) theory of the reals' in the following sense. Any true claim about the reals has a corresponding *-transform true claim about the hyperreals; and any true claim about the hyperreals *that can be represented as the *-transform of a claim about the reals* has a corresponding true claim (that claim of which it is the *-transform) about the reals. Ballard aptly describes such a transfer principle as insuring that a nonstandard theory "is 'safe' for conventional mathematicians" (Ballard [1], p. 77). In particular, the transfer principle supplies a new, and sometimes useful (or elegant, or interesting, or relatively more simple), way of proving theorems about the real numbers.[10]

The *-transforms of first-order propositions about the reals (propositions that involve quantification over only the real numbers themselves) have essentially the same 'sense' as the first-order propositions about the reals of which they are the *-transforms. Thus, for example, corresponding to the claim that, for every positive real number (element of R), there is a smaller positive real number, there is a straightforward analogous claim (*-transform) about positive hyperreal numbers (elements of *R). However, while higher-order propositions about the reals have true *-transforms, they exhibit a more marked 'shift in meaning'. To consider a concrete example, the ordered field of reals is 'complete' in the sense defined above: every nonempty subset $X \subseteq R$ that has an upper bound has a least upper bound (supremum). This characteristic can be formulated as a second-order truth about the reals—second-order because it involves quantification over sets of reals or, in terms of our restriction to bounded quantifiers, over elements of the power set of the reals, $\mathcal{P}(\mathrm{R})$. We face here what initially appears to be an antinomy. There is a true *-transform, that is, a 'corresponding' truth about the hyperreals, to the second-order proposition expressing the completeness of the ordered field of reals. But, because the ordered field of hyperreals is non-Archimedean, there will be sets of hyperreals (elements of $\mathcal{P}(^*\mathrm{R})$) which are bounded above but have no *least* upper bound (e.g., among many other sets, the set of positive infinitesimal hyperreals). The antinomy is only apparent because the bounded second-order quantifier in the *-transform of the completeness proposition rangers over the *-transform of the power set of the reals, that is, $^*\mathcal{P}(\mathrm{R})$, and this set is not identical to (is, in fact, a *proper* subset of) the power set of the hyperreals, that is, $\mathcal{P}(^*\mathrm{R})$.

To speak informally, there are entities of the hyperreal superstructure $V(^*\mathrm{R})$—

here, certain sets of hyperreals or elements of $\mathcal{P}(^*\mathrm{R})$—that the *-transforms of truths about the reals fail 'to detect' or 'to know about'. These are just the entities that would 'make trouble'—in our particular example, sets of hyperreals or elements of $\mathcal{P}(^*\mathrm{R})$ that, while bounded above, do not have a least upper bound. This situation is the manifestation of an important distinction among three kinds of entities of the hyperreal superstructure $V(^*\mathrm{R})$. A *standard* entity y is an element of $V(^*\mathrm{R})$ that is the image, in terms of the monomorphism *, of some x of $V(\mathrm{R})$ (i.e., $(\exists x \in V(\mathrm{R}))(y = {}^*x)$). An *internal* entity y is an element of $V(^*\mathrm{R})$ that is an element of some standard entity x that is itself an element of $V(^*\mathrm{R})$ (i.e., $(\exists z \in V(\mathrm{R}))(\exists x \in V(^*\mathrm{R}))(x = {}^*z \wedge y \in x)$). It is easy to show that all standard entities are internal ones, but the converse does not hold. Finally, an *external* entity is an element of $V(^*\mathrm{R})$ that is not internal. Propositions that involve essential reference to external entities constitute that problematic part of the language/theory of hyperreals to which I earlier alluded: the part that does not 'correspond' to the language/theory of the reals.

It is only internal entities that bounded quantifiers of the (syntactic) *-transform of our language/theory of reals range over. Or, from our original semantic perspective, it is these entities in the domain $V(^*\mathrm{R})$ that the theory of reals, *when interpreted in the nonstandard model* $\langle V(^*\mathrm{R}), \in\rangle$, 'detects' or 'knows about'. The theory of reals so interpreted does *not* detect/know about external entities, which thus have a sort of 'ghostly' presence in the model. I shall later return to the philosophical implications of this ghostly presence. For the moment, I merely note some straightforward but perhaps somewhat surprising consequences of the distinction among standard, internal, and external entities. The *-image $^*r$ of each and every real number r (member of R) is a standard entity; and, indeed, such images are customarily thought of as simply being the real numbers 'embedded' into the set *R of hyperreals. However, *R contains infinitesimal and infinitely large elements that are not the *-images of any real numbers; hence such infinitesimal and infinitely large hyperreals are not standard but *are* internal (because they are members of the standard entity *R). However, the *set* $^\sigma\mathrm{R} = \{^*r \in {}^*\mathrm{R}\colon r \in \mathrm{R}\}$ of all and only the 'embedded' standard reals is an external set—as is the set of all and only infinitesimal elements of *R and the set of all and only infinitely large elements of *R. Consequently, we can say that, although the theory of reals, when interpreted in the nonstandard model $V(^*\mathrm{R})$, detects or knows about the 'embedded' or standard reals 'individually' or 'distributively', it has 'lost sight' of the *set* of standard reals. In fact, it turns out that any *infinite* set containing all and only standard reals is external (Stroyan and Luxemburg [28], pp. 56–7, Theorem 4.5.3). One might say that, with respect to the interpretation of the theory of reals within the nonstandard model $V(^*\mathrm{R})$, the standard reals become 'indistinguishable' from nonstandard hyperreals in infinite sets. As we shall later see, a generalization of this result holds for nonstandard or 'internal' set theory and is utilized to obtain a striking 'philosophical' result by McLaughlin.

3.2 Nonstandard set theories One of the two first developments of a foundation for nonstandard mathematics that is 'strictly set-theoretic' (as opposed to model-theoretic) was the axiomatic 'internal set theory' (IST) of Nelson, dating from the late 1970s (Nelson [22]).[11] Nelson constructs IST by beginning with Zermelo-Fraenkel set theory with the axiom of choice (ZFC) and adding

(a) a new 1-place predicate ('is standard') and
(b) three new axiom schemata governing the new predicate.

A well-formed formula (wff) of IST is called 'external' if it contains the new predicate 'is standard' and is called 'internal' if it does not. In practice, Nelson typically uses the 'is standard' predicate to produce restricted quantifiers:

'$(\forall^{\mathrm{st}}x)\varphi$'	is shorthand for	'$(\forall x)(x$ is standard $\longrightarrow \varphi)$',
'$(\exists^{\mathrm{st}}x)\varphi$'	for	'$(\exists x)(x$ is standard $\wedge \varphi)$', and
'$(\forall^{\mathrm{st\ fin}}x)\varphi$'	for	'$(\forall x)([x$ is standard $\wedge x$ is finite$] \longrightarrow \varphi)$',

and so on, (where 'x [a set] is finite' has its usual meaning: "it is an abbreviation for the internal formula which asserts that there is no bijection of x with a proper subset of itself" [22], p. 1166). The three added schemata are as follows:

(T [Transfer]) $(\forall^{\mathrm{st}}t_1)\cdots(\forall^{\mathrm{st}}t_k)[(\forall^{\mathrm{st}}x)A(x, t_1, \ldots, t_k) \longrightarrow (\forall x) A(x, t_1, \ldots, t_k)]$, where $A(x, t_1, \ldots, t_k)$ is an internal wff with free variables $x, t_1, \ldots, t_k$ and no other free variables;

(I [Idealization]) $(\forall^{\mathrm{st\ fin}}z)(\exists x)(\forall y \in z)B(x, y) \equiv (\exists x)(\forall^{\mathrm{st}}y)B(x, y)$, where $B(x, y)$ is an internal wff with free variables x, y (and possibly other free variables);

(S [Standardization]) $(\forall^{\mathrm{st}}x)(\exists^{\mathrm{st}}y)(\forall^{\mathrm{st}}z)[z \in y \equiv (z \in x \wedge C(z))]$, where $C(z)$ is either an internal or external wff with free variable z (and possibly other free variables).

Of these schemata, (T) and (I)—as their designations indicate—provide the axiomatic means for procuring analogues of those desirable characteristics of nonstandard universes that I earlier discussed: transfer (or bounded elementary embedding) and saturation (or idealization), respectively. The attempt to form sets from *external* wffs using the 'normal' ZFC abstraction or 'separation' schema (*Aussonderung Axiom*) is prohibited as "illegal set formation" and, in fact, quickly leads to antinomy. The standardization schema (S) provides a very restricted substitute: one can use a wff—internal or external—to form a (standard) set *whose standard members* satisfy that wff if one 'applies' that wff to a set that is standard.

A salient feature of Nelson's IST, which both contributes to its elegance and has dissuaded many mathematicians from accepting it as an adequate set-theoretic foundation for nonstandard mathematics, is that it does not allow for the existence of any sets that, in different formulations (such as the nonstandard-universe approach or the ultrapower construction that we have considered), are termed *external.* Thus, when IST is used to construct nonstandard hyperreal mathematics in a way analogous to which ZFC can be used to construct the mathematics of the real numbers, there will be no set of all and only *standard* natural numbers, or sets of *standard* rational or real numbers. Nor will there be, for example, any sets containing all and only the (nonstandard) infinitely large hyperreal numbers or 'hypernatural' numbers, or any set of all and only the (nonstandard) positive infinitesimals, or any of the sets that nonstandard mathematicians have come to call '*monads*'—for each hyperreal (standard or nonstandard) number r, the set (equivalence class) of all hyperreal numbers s that are at an infinitesimal distance from r (i.e., $\{s : s \simeq r\}$). From the perspective of IST, such sets have, as it were, 'disappeared'. Thus, the set of natural numbers ℕ in IST is a set containing many nonstandard members: in fact, it can be depicted as being constituted as an ordered set beginning with the standard naturals followed by a *densely ordered* collection (without first or last member) of discretely ordered sets of numbers, each with the order type of the signed integers (that is, in Cantorian terms, the order type of the hypernatural ℕ is $\omega + (\omega^{*}{+}\omega)\theta$, where θ is a dense order type without first or last element). Although the *standard* natural numbers may be said to be, individually, members of this set, the set of all and only these standard members (which we would intuitively like to think of as constituting the initial 'ω-segment' of the ordered set ℕ) simply does not exist in IST—as a member of the power set of ℕ or in any other way.

From a purely mathematical point of view, IST's elimination of external sets has generally been viewed as at least awkward. Fletcher comments that "external sets, such as monads, galaxies, and the set of standard elements of a standard set S (written as ${}^{0}S$), are very common in nonstandard arguments, and to express such arguments in IST requires rephrasing to remove all references to external sets" (Fletcher [10], p. 1001).[12] He adds that

> it seems probable that all external sets introduced in proving standard theorems (as opposed to external sets used for studying nonstandard models for their own sake) can be defined in terms of internal sets and the predicates 'standard' and 'internal'; so the above paraphrasing should always be possible. Still, the resulting statement of IST is much less transparent and more awkward to handle. ([10], p. 1004)

IST's 'disappearing' of external sets, in addition to producing mathematical 'awkwardness' (for those mathematicians who wish to use it in "proving standard theorems" in ZFC, real analysis, etc.), can also lead to a rather peculiar picture of mathematical ontology. In an engaging and accessible presentation of Nelson's IST, Robert succinctly sets forth one such picture:

> . . . the set ℕ of natural numbers is the same in NSA [i.e., Nelson's IST] simply because it is unambiguously defined in (ZF), and thus is part of our new system. More explicitly, we are *not going to add elements to the classical set* ℕ *of natural numbers*, and we shall never refer to an 'extension' *ℕ of ℕ as Robinson initially did. But if ℕ still represents the same classical set, it is also true that the new deduction principles—resulting from the new axioms—may give a psychological feeling of extension since they reveal elements that were unknown to (ZF). In a sense, the new axioms bring to life unsuspected elements in the traditional set ℕ. While this set ℕ has not changed, people working with NSA discern 'more elements' in it, because they have a richer axiomatic. Of course, these unsuspected elements had always been there . . . . (Robert [24], p. 9)

Most developments of nonstandard mathematics have been Janus-faced, to use the term that I previously introduced. What does this mean? A nonstandard universe, for example, consists of a standard, intended model, and a nonstandard model. So it looks *both* 'back' to the standard, intended interpretation or model of (a part of) the mathematical theory of the real number system *and* 'forward' to a nonstandard model of that same theory. To oversimplify a bit, the standard entities and internal entities of these developments represent areas where the two models 'hook up' (in terms of the monomorphism * from the standard to the nonstandard models); external sets represent areas (entities of the latter, nonstandard model) where they do not 'hook up'. If we choose to transform * into a syntactic mapping in order to create two 'theories', a standard one and a nonstandard one, the later becomes a conservative extension of the former. But, whichever perspective we adopt, it is characteristic of most developments of nonstandard mathematics to keep in view (and ready for use) *both* models or theories-*cum*-models. This is what I mean by 'Janus-faced'.

Nelson's elimination of external sets in the IST suggests a point of view that is *not* Janus- faced in this sense but exclusively 'forward' looking: in effect, the standard model (theory-*cum*-model) has been *replaced* by a nonstandard one. The question then arises as to what has happened to the various (infinite) sets of the original standard model. In most developments of nonstandard mathematics they survive, after a fashion, as external sets of the nonstandard model (theory-*cum*-model). But, since Nelson's IST has no external sets, they cannot survive in that fashion. What, then, has happened to such sets in IST? Robert's answer seems to be that they were never there to begin with! Because of our relatively impoverished 'axiomatic' (of a standard set theory such as ZFC) we simply did not recognize all the elements present in, say, the set ℕ of natural numbers or the ordered field ℝ of reals—but the unrecognized elements were actually there all along. Other comments by Robert indicate that he may not, in fact, wish to be committed to the raging mathematical platonism that would seem to accompany such a perspective. However, the passage that I quoted earlier suggests, I believe, that—at least as Robert interprets it—Nelson's IST represents an approach to the hyperreals that is significantly different from the more usual Janus-faced perspective, an issue to which I return in the 'philosophical' comments of the following section.[13]

The objections by mathematicians to Nelson's IST, however, have typically been more 'practical'. Nonstandard mathematics has for the most part been developed as a conservative extension of standard mathematics: that is, it has quite self-consciously been developed in such a way that whatever theorems provable in nonstandard mathematics about 'standard' (set-theoretic, real-analytic, etc.) mathematical entities are, in principle, provable in the relevant branch of standard mathematics (ZFC, real analysis, etc.). I have already rehearsed Fletcher's comments about the 'awkwardness' of IST's elimination of external sets, which are often used in nonstandard arguments. There is perhaps some irony in the fact that Nelson himself introduces external sets as a convenience, while commenting that "external sets are not entities of IST" ([22], p. 1176). Chang and Keisler note that "a disadvantage [of IST] is that the language cannot talk about sets of external sets [it *can* 'talk about' particular external sets whose elements are internal by use of wffs containing the 'is standard' predicate], such as the σ-algebra generated by an algebra of internal sets. In practice, internal set theory has been adequate for certain areas of Robinsonian analysis (e.g., singular perturbations), but inadequate for others (e.g., probability theory, Banach spaces)" ([3], p. 287). Consequently, subsequent work on formulation of a set-theoretic foundation for nonstandard mathematics has provided for the existence of external sets.[14]

4 Some philosophical considerations After the technicalities of the preceding section, I return to one of the principal conceptual questions of the paper: Is the nonstandard method of according numerical status to incomparables—by means of the hyperreal extension of the field of real numbers—a simple and straightforward analogue of the nineteenth-century method of according numerical status to incommensurables—by means of the real extension of the field of rational numbers? As I have now indicated, there certainly are some similarities, beginning with an informal picture of the number line. The 'construction' of the reals by Dedekind and Cantor suggests the filling in of 'gaps' in the rational line by irrational numbers. Similarly, according to a common picture of the hyperreals, infinitesimals 'fill in the gaps' surrounding each standard, finite real (yielding monads); and the negative and positive directions of the real line are extended by 'galaxies' of hyperfinite reals. In his elementary calculus text Keisler [15], which is based on Robinson's development of nonstandard analysis, Keisler uses the heuristic devices of the 'infinitesimal microscope' and the 'infinite telescope' for 'looking at', for example, the behavior of the slope of a 1-place function within the 'monadic neighborhood' of a given ordered pair of points. In fact, these devices are formally defined in [16].

From a somewhat more rigorous perspective, there is an obvious similarity between the Cantorian construction of the reals by means of Cauchy sequences and the construction of the hyperreals by means of ultrapowers. However, we saw that the field $\mathbb{R}$ of reals numbers is the unique complete ordered field up to isomorphism. Since the non-Archimedean field $^*\mathbb{R}$ of hyperreals is not isomorphic to the reals, it is, of course, not complete in the technical sense that we earlier noted: every nonempty subset $X \subseteq \mathrm{R}$ that is bounded above has a least upper bound. For example, the (external) set of infinitesimals, while bounded above (by any standard real), has no supremum: there is no greatest infinitesimal nor smallest standard real number. An analogous uniqueness theorem for the hyperreal number system is more difficult to construct because of the fact that nonstandard models of the theory of real numbers—such as the hyperreal nonstandard universes $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$—of arbitrarily large cardinality can be found. However, it turns out that, if one desires, one can find such a uniqueness theorem: If we require (i) that the monomorphism * of a nonstandard universe $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ satisfies the transfer principle, (ii) that $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ be saturated over $V(\mathrm{R})$, and (iii) that both $^*\mathrm{R}$ and the set of all internal sets have the cardinality of the first uncountable inaccessible cardinal,[15] there is, up to isomorphism, a unique such nonstandard universe $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ (see [16]). However, it might be objected that condition (iii) seems to be quite arbitrary.

A different perspective—one that does not make platonist assumptions about the foundational, 'givenness' of $\mathbb{R}$—begins with the observation that different set theories (e.g., one in which the continuum hypothesis holds and one in which it does not) might yield nonisomorphic $\mathbb{R}$s. One could then infer from a perspective such as that of Feferman ("I am convinced that the Continuum Hypothesis is an inherently vague problem that no new axiom will settle in a convincingly definite way," [9], p. 109) that $\mathbb{R}$ itself is not a unique or determinate, well-defined mathematical object. Consequently, any 'ambiguity' or 'vagueness' with respect to the ontological status of $^*\mathbb{R}$ is matched by that of $\mathbb{R}$ itself.[16]

My conclusion is that there is not a sufficiently obvious, clear, and determinate sense of 'similar' in order to supply a plausible *mathematical* answer to the question of whether the extension of the ordered field of reals to that of hyperreals is similar to the extension of the ordered field of rationals to that of reals. Similarity here appears to be largely in the eye of the beholder; and whether one finds it or not will depend upon any number of different predilections, assumptions, and commitments.

However, I wish to return to a fundamental difference between the classical nineteenth-century approach to incommensurables and the influential twentieth-century nonstandard approach to incomparables. As the term 'nonstandard' suggests, nonstandard hyperreal models were developed as alternative, nonstandard models of the *theory of real numbers*. A principal motivation for the development of nonstandard analysis was, in the words of Robinson, the conviction "that the theory of certain types of non-Archimedean fields can indeed make a positive contribution to classical Analysis" ([25], p. 261). Even for those applications of nonstandard mathematics that essentially employ nonstandard models (such as Loeb measure on a hyperfinite grid, concerning which see Section 5.2 below of the present paper), it has generally seemed necessary or advisable to mathematicians to establish a reference to some established area in classical Archimedean mathematics (such as Lebesgue measure on the real line in the case of Loeb measure) (see Keisler [17], p. 222).

So it is perhaps not surprising that most developments of nonstandard mathematics have had the Janus-faced character to which I earlier alluded: they look *both* 'back' to the standard, intended interpretation or model of (a part of) the mathematical theory of the real number system *and* 'forward' to a nonstandard model of that same theory. This Janus-faced character is made particularly perspicuous by the formal representation $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ of a nonstandard universe, which contains both a superstructure on the reals and a superstructure on the hyperreals, together with the monomorphism * connecting them. It is clear that, at least with respect to infinite sets, the two superstructure models are incompatible in the sense of assigning infinite sets that are not isomorphic as extensions of the relevant predicates— for example, $\mathbb{N}$ and $^*\mathbb{N}$ to 'is a natural number, $^*\mathrm{R}$ and R to 'is a real number', similarly for 'is finite', and so on. As we saw, *within the nonstandard model*, the former infinite sets of standard elements possess only a ghostly existence as external sets, which the bounded quantifiers do not recognize: a ghostly but important existence, however. If the principal value of such nonstandard models is to serve as one element, among many, in the tool kit of mathematicians working in classical analysis or some other area of Archimedean mathematics, it is crucial not to let any of the entities of the original intended Archimedean model—or other important external sets of the nonstandard model—disappear.

The situation with respect to the nineteenth-century construction of the reals from the rationals was quite different. It certainly was never supposed that the (Archimedean) ordered field $\mathbb{Q}$ of rationals and the (Archimedean) field $\mathbb{R}$ of reals were alternative—and in a sense incompatible—models or interpretations of the *same* mathematical theory. The added irrational elements of the latter were simply numerical representations of entities that had possessed, since at least Greek antiquity, a well-established and respectable geometrical presence—whereas the numerical representation of incomparables (infinitesimals and infinitely great but still 'hyperfinite' reals) added by hyperreal models have never had such a well-established and respectable presence. And there never seemed to be any danger that, in working with the field of reals as opposed to the field of rationals, mathematicians had to worry about 'losing' any structures thought to be mathematically significant. Therefore, there was no compelling *mathematical* reason for the nineteenth-century real analyst (or 'rational analyst') to adopt a Janus-faced perspective, keeping in view both ordered fields $\mathbb{Q}$ and $\mathbb{R}$ as alternative models for the 'theory of rational numbers'. One can certainly, if one wishes, interpret the theory of rationals in the latter, 'enlarged' model, that is, the field $\mathbb{R}$ of reals. But, in doing so, it does not seem that one has 'lost' anything significant about the original intended model $\mathbb{Q}$ in the way in which one has 'lost' the original set $\mathbb{R}$ of reals—it has either become an external set or has disappeared altogether—in the nonstandard model of the theory of reals, that is, the superstructure $V(^*\mathrm{R})$.

While it is eminently useful in terms of most current mathematical employments of nonstandard models, the Janus-faced perspective introduces, as I shall argue in the next section, a sort of ambivalence into the attempt to use nonstandard models *conceptually*—for example, the attempt to use such models in order to resolve certain classical problems concerning continua.

5 The philosophical employment of nonstandard models: three examples In this last section, I consider three attempts to apply concepts of nonstandard analysis to several of Zeno of Elea's famous puzzles concerning continuous motion. I shall argue that what I have termed the Janus-faced propensity of contemporary nonstandard mathematics imparts to these attempts to employ nonstandard concepts philosophically a certain ambiguity—a '*sic et non*' character (rough English translation: "well, yes, but . . . ").

5.1 Zeno's arrow paradox In 1982, in White [29] I applied nonstandard concepts to resolve a version of Zeno's Arrow paradox formulated by Jonathan Lear:

1. Anything that is occupying a space just its own size is at rest.
2. A moving arrow, while it is moving, is moving in the present.
3. But in the present, the arrow is occupying a space just its own size.
4. Therefore, in the present the arrow is at rest.
5. Therefore, a moving arrow, while it is moving, is at rest (Lear [19]).

The most common resolutions of the Arrow paradox, in this form, would probably deny one or both of premises 1 and 2. If 'the present' is a durationless temporal instant or 'point', Aristotle will deny 2 (and, by implication, some instances of 1), responding that the arrow is neither at rest or moving with respect to such a 'now' (τό νῦν) because both motion and rest imply a lapse or duration of time. Some contemporary resolutions will deny 1 (and perhaps 2) by introducing a 'derivative' sense of motion (pun intended) applicable to the temporally instantaneous states of bodies. According to such a conception, although motion in the full-blooded sense implies temporal lapse or duration, a body undergoing such motion and possessing positive instantaneous velocity with respect to a temporal point or instant can correctly be said to be moving 'with respect to' (or 'at', but not 'in') such a 'now'.

However, some ancients—perhaps some Stoics—may have been inclined to regard the present or temporal now (τό νῦν) as having duration. From such a perspective, as Lear suggests, premise 3 can be attacked "by developing a theory of time in which the present can be conceived as a period of time. . . . one can then proceed, as Aristotle did not, to give a sense to the notion of an object moving at an instant or at the present instant [in the 'full-blooded' sense of moving]" ([19]). I suggested, in effect, that temporal presents, instants, or nows be identified with monads of the hyperreal line used to represent the temporal dimension of motion ([29]). Recall that a monad is an equivalence class of points standing at infinitesimal distances from each other: for a given $t\{t' : t \simeq t'\}$. This, so to speak, is to give temporal instants infinitesimal duration, making them 'fat'. I could then diagnose the argument as a fallacy involving equivocation on the phrase "occupying a space just its own size," which can mean, for something of 'size' r, either (a) "occupying a space r' such that $r = r'$" or (b) "occupying a space r' such that $r \simeq r'$." Sense (a), but not (b), is taken to make premise 1 true; while sense (b), but not (a), makes premise 3 true ([29], pp. 242–3).

While such a resolution possesses a certain neatness and cleanness, it depends on ascribing to monads (which are external sets not even existing in Nelson's IST) a 'physical' or temporal reality—as 'nows' or temporal instants—that is withheld from the hyperreal points contained in such monads. Since each such monad contains precisely one standard real point, it is possible to view such monads as the 'fat surrogates' of points of the real-line representation of the temporal dimension of motion. So my analysis 'looks backward', in effect, to the (external-set replacements of the) points in the real-representation of the temporal dimension of motion in order to obtain its 'basic units' of time. Yet, it 'looks forward' to the hyperreal line for its representation of motion: motion is change of place with respect to lapse of time where both 'change of place' and 'lapse of time' are defined in terms of *hyperreal coordinates*. Consequently, a hyperreal version of the Arrow paradox may be formulated simply by identifying 'the present' or a 'now' with any hyperreal point, rather than with any monad. And it is clear that such a 'strengthened Arrow' cannot be resolved by the same maneuver; its resolution apparently will depend on one of the 'classical' maneuvers, such as denying premises 1 or 2.

5.2 Zeno's dichotomy paradox: one resolution According to Zeno's Dichotomy paradox, a runner is charged with traversing a distance of unit length. But, according to Zeno, before the runner can reach the goal point—call it '1'—he must first traverse, in order, an infinite sequence of checkpoints $\{(2^n - 1)/2^n\}$, beginning with $n = 1$ and continuing as n increases without limit, that is, $\{1/2, 3/4, 7/8, 15/16, 31/32, \ldots\}$. According to one (charitable) interpretation, Zeno's principal point is that such a runner cannot reach the goal 1 because he *first* has to complete, sequentially, an infinite number of actions (each one associated with reaching a checkpoint and no one of which is equivalent to reaching the goal 1) of which there is no last member—and that, Zeno believes, is impossible. According to another version of the Dichotomy, the sequence of tasks of reaching checkpoints is inverted, $\{\ldots 1/32, 1/16, 1/8, 1/4, 1/2, 1\}$, so that there is no first member of the sequence and the runner, according to Zeno, cannot 'get started'.

Keisler outlines a resolution of the Dichotomy that makes use of a nonstandard concept to which I have previously alluded, a *hyperfinite* grid. Where H is an infinite hypernatural number, the hyperfinite grid $\mathbb{H}$ with mesh $1/H$ is the set of all multiples of $1/H$ between $-H$ and H. As Keisler notes, an H is usually selected such that every standard natural number divides it (e.g., by letting $H = J!$ or J factorial, for some infinite hyperfinite number J), with the result "that each standard rational number belongs to $\mathbb{H}$, that is $\mathbb{Q} \subseteq \mathbb{H}$." ([17], p. 219). Keisler applies such a hyperfinite grid, restricted to the unit interval [0, 1], to Zeno's Dichotomy:

> On the hyperfinite grid, Zeno's [Dichotomy] Paradox is resolved as follows. We can get from 0 to 1 in H steps by taking one step of length $1/H$ every $1/H$ seconds, always staying in the hyperfinite grid $\mathbb{H}$. Along the way, we will pass through all the points $1/2, 3/4, 7/8$, and so on, since they all belong to the set $\mathbb{H}$. Of course, we will overshoot irrational points such as $\sqrt{2}/2$, but there will be a time at which we pass from below $\sqrt{2}/2$ to above $\sqrt{2}/2$ with one step of length $1/H$. ([17], p. 233)

Such a hyperfinite grid, as well as its restriction to the unit interval, are 'hyperfinite'—that is, they belong to the extension of the predicate 'is finite' when interpreted in the nonstandard superstructure model $V(^*\mathrm{R})$ of hyperreal numbers. As Keisler notes, such a hyperfinite subset of $^*\mathbb{R}$ "inherits the first order properties of finite subsets of $\mathbb{R}$" ([17], p. 218) as well as appropriately weakened higher-order properties. Thus, for example, there is no *internal* one-to-one mapping between such a set and any of its proper subsets. Also, in terms of the < relation, there is a 'first' member and a 'last' member of each such set, and its members are discretely ordered: that is, any member that has a successor (predecessor) has a unique, immediate successor (predecessor). Consequently, many of the worries about the possibility of completing, sequentially, an infinite sequence of tasks seem to disappear. Although from an 'external' (standard) point of view, the runner must complete an infinite sequence of tasks, there will be a first task and a last task for him to complete; and any task strictly between these will have a unique, proper succeeding and a unique, proper preceding task, and so on. And, as Keisler notes, each of the Zenonian checkpoints in the sequence $\{(2^n - 1)/2^n\}$ is 'embedded' as one of these tasks—as is each of the checkpoints in the inverted Zenonian sequence $\{\ldots 1/32, 1/16, 1/8, 1/4, 1/2, 1\}$.

Does, then, Keisler's resolution 'work'? *Sic et non.* There are subsets of a hyperfinite grid of a given mesh (as restricted to [0, 1]) that raise some of the same Zenonian worries about completing sequentially an infinite sequence of tasks. For example, there is the set of all initial steps, individually of length $1/H$, the sum of which is *less than* any real value, as well as the complement of this set relative to the hyperfinite grid restricted to [0, 1]. It would seem that all of the tasks or steps in the former set would have to be completed, sequentially, before the runner could undertake work on the tasks/steps in its complement. But there is no last task/step in the former set nor any first task in the latter. These 'problem-causing' subsets, however, must be *external* subsets of the hyperfinite grid restricted to [0, 1]. This means that they do not fall within the range of the higher-order bounded quantifiers over sets of hyperreals (that is, quantifiers ranging over $^*\mathcal{P}(\mathrm{R})$), when the wffs of the theory of the reals are interpreted in the nonstandard superstructure $V(^*\mathrm{R})$. Does it also mean that the sets are 'not there' to cause problems for Keisler's analysis of the Dichotomy? Well, in most formulations of nonstandard analysis such problem-causing sets *are* 'there', as external sets or members of $\mathcal{P}(^*\mathrm{R}) - {}^*\mathcal{P}(\mathrm{R})$. Although this may be a sort of mathematically 'ghostly' existence (in terms of the interpretation of the theory of reals in the nonstandard model $V(^*\mathrm{R})$, or in terms of that 'part' of the 'theory of hyperreals' pertaining to embedded reals), it is far from clear (to me, at least) that the existence of such subsets does not raise more or less the same issues about the possibility of completing, sequentially, an infinite sequence of tasks, that Keisler's resolution of the Dichotomy in terms of hyperfinite grids was supposed to avoid.

5.3 Zeno's dichotomy paradox: another resolution Perhaps the most interesting attempt with which I am familiar to apply nonstandard concepts to a classical philosophical problem is the 'critical' application of Nelson's IST to Zeno's Dichotomy paradox by McLaughlin and Miller. The argument of McLaughlin and Miller depends on an epistemological assumption:

> (E2) The fact that an object is located at a point in spacetime cannot be established if the coordinates describing the point are nonstandard real numbers. (McLaughlin and Miller [21], p. 378)

McLaughlin and Miller add that "the phrase '[t]he fact that' in E2 means that the object's location has been observationally verified or could have been observationally verified had one been sufficiently equipped and attentive to capture the requisite numerical description of the event" ([21], p. 379). In a later paper by McLaughlin, this assumption becomes the "critical mensuration thesis," which he characterizes as follows:

> every phenomenon can be completely described through the use of real numbers, but not all real numbers can be used for describing phenomena. The first clause, the "mensuration thesis," in the statement of the greater thesis, rests upon the success of experimental science. The second clause must be argued, and this is carried out through the medium of internal set theory . . . . (McLaughlin [20], p. 283)

Since McLaughlin is working within the framework of Nelson's IST, which does not provide for the existence of external sets, the extension of the predicate 'real number', as he uses the phrase, is what is designated $^*\mathrm{R}$ in other formulations, that is, the hyperreal numbers, and includes nonstandard infinitesimals, infinitely large reals, and so on. As it turns out, the second clause will rule out precisely the nonstandard reals for use in describing physical processes:

> Although the mensuration thesis has appealed to real numbers for the means to express results of a measurement process, it is clear that the thesis can be extended to other mathematical objects which might serve as measurement labels, for example, complex numbers, vectors, real intervals. Specimens of such objects would be suitable candidates for the process only if they were standard. ([20], p. 289)

And, says McLaughlin, "we consistently adopt the perspective of an observer who is measuring phenomena and have shown that nonstandard numbers are not available as measurement labels for those phenomena" ([20], p. 289).

Now, consider the set of 'checkpoints' of Zeno's Dichotomy, designated by McLaughlin and Miller as $C = \{r \in [0, 1] : r = 1 - 2^{-j}, 1 \leq j < \infty\}$ ([21]). From the perspective of IST, this (standard) set will contain (a great many) elements that are nonstandard. That is, it will include infinitesimals for (all) values of $r = 1 - 2^{-j}$ where j is what is variously called an 'infinite hypernatural' or 'illimited [positive] integer' (see Robert [24], p. 17)—which numbers possess, of course, the first-order properties of standard *finite* natural numbers. But, then, by the critical mensuration thesis (hereafter, CMT), the set will contain (nonstandard) elements that cannot describe any physical process and, consequently, no runner could correctly be described as traversing the set C of spatial loci.

But what of the set frequently characterized, in some formulations of nonstandard set theory, as $^{\sigma}C$ (or ^{0}C)[17], the set of all and only the *standard* elements of the set C? We might—perhaps with some prejudice—characterize this as the 'original set giving rise to the Zenonian worry'. In other formulations of nonstandard mathematics it would be an external set. But in Nelson's IST, although all of its members can be said to exist *individually* as standard mathematical entities, the set $^{\sigma}C$ simply does not exist. In fact, IST does not allow for the existence of *any* set that is infinite (in the usual, standard sense of 'infinite') but contains only standard elements (see [22], p. 1167, Theorem 1.1).

So, the supposed infinite sequence of tasks of Zeno's Dichotomy cannot be described by the (nonexistent) set $^{\sigma}C$; and application of the CMT rules out its description by the set C, which contains nonstandard members. Consequently, McLaughlin draws a finitistic conclusion about (the mathematical description of) a physical process such as Zeno's Dichotomy:

> For Zeno's Dichotomy, incursion into an infinitesimal neighborhood of 1 was seen to be possible but epistemologically opaque, disabling his claim of paradox. . . . Assume that there are no physical constraints to prevent a traverse of any finite segment of the Checkpoint sequence of The Dichotomy, and . . . implement a counting scheme to register the passage of the moving object past each checkpoint. It must be the case that the count ceases prior to the recording of an unlimited natural number. This implies that the count must terminate at some standard natural number . . . ; to avoid paradox, the premise of no-physical-constraints must be judged false. That is, phrased positively, *physical* reasons must prevent the observations from being made. ([20], pp. 290–91)

McLaughlin's argument, perhaps somewhat oversimplified, is the following: Physical-epistemological considerations preclude any mathematical description of a physical process that appeals to a set containing nonstandard numbers. But IST provides only for *finite* sets of elements containing nothing but standard elements. Therefore, the mathematical description of physical processes must be finitistic—apparently, suggests McLaughlin, because of physical-epistemological reasons.

McLaughlin does not intend the CMT to yield such a finitistic conclusion *directly*. The use of IST—and in particular, its 'disappearance' of external sets—is a crucial step in his argument. Consequently, it seems to me that one cannot infer that the finitistic conclusion of his argument is due *exclusively* to physical-epistemological considerations. It is the *combination* of the CMT and IST that yields such a conclusion. If one were to accept IST + CMT, that fact would entail that any physical theory could not fail to be finitistic, "on pain of being mathematically unintelligible," as McLaughlin has, in personal communication, expressed it to me. However, I suspect that the appeal of the finitism characteristic of physical theories—and, indeed, the persuasiveness of arguments on behalf of such a characteristic of physical theories—would rest on considerations quite independent of IST + CMT.

The fact that Nelson's IST is, in its elimination of external sets, less Janus-faced than competing formulations—including other set-theoretic formulations—of nonstandard mathematics seems to be crucial to McLaughlin's argument. From the perspective of alternative such formulations providing for the existence (as an external set) of the 'original' Zenonian set $^{\sigma}C$ of checkpoints, there would be no apparent reason to disallow this set as an acceptable description of a sequence of actions performed by Zeno's runner, since McLaughlin's CMT *in itself* does not rule it out. But then we are faced afresh with all the puzzles concerning the sequential performance of an infinite sequence of acts. One could, of course, invoke physical-epistemological considerations in order to strengthen the CMT so that it *does* commit one to finitistic descriptions of physical processes. But that would be to render the IST otiose in the argument for such a conclusion.

To summarize, I think that any arguments on behalf of some form of finitism are going to be independent of Nelson's IST. McLaughlin's combintation of CMT + IST is presented in such a way that it may look like it is intended as an argument for such finitism. However, I think that what McLaughlin really intends is to *model*, mathematically and epistemologically, such a physical finitism. IST, among the various mathematical developments of nonstandard mathematics, is singularly useful for such an enterprise because, in dispensing with *all* external sets, it lacks the Janus-faced character of virtually all other contemporary developments of nonstandard mathematics. However, this very fact has rendered IST unsuitable, in the view of many mathematicians, for many of the most interesting mathematical uses that have been thus far found for nonstandard mathematics.

6 Conclusion It is obvious, I think, that incommensurable magnitudes have found a secure numerical home in the real line, a process that was effectively concluded, from a mathematical point of view, in the nineteenth century and that involved a 'paradigm shift' with respect to the classical Greek treatment of incommensurables.

Although the development of nonstandard mathematics has provided incomparables with a spacious numerical *Lebensraum* in the hyperreal line, I believe that it is clear that there has not yet been a similar paradigm-shift with respect to incomparable magnitudes: that is, there is not yet any widely accepted arithmeticization of incomparable magnitudes in the form of a paradigm-shift from real to hyperreal line. Although there are now some important exceptions (e.g., Loeb measure, canards), contemporary nonstandard mathematics has, for the most part, been developed as a tool for doing classical, Archimedean mathematics. Consequently, it has been essential to this perspective not to abandon classical Archimedean models while, at the same time employing nonstandard models as needed. The result has been the Janus-faced character of contemporary nonstandard mathematics, which militates against a shift away from the Archimedean paradigm.

In my view this Janus-faced character has made nonstandard mathematics of doubtful use, at present, in addressing 'deep' conceptual or philosophical issues pertaining to continuity and infinity. Nonstandard mathematics has, in effect, adopted a sort of ontological ambivalence toward external sets of nonstandard models (and set theories). As I have attempted to show, this fact raises problems for attempted nonstandard resolutions of classical conceptual problems pertaining to continuous magnitudes.

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NOTES

1. I reserve my technical term 'incomparable' for magnitudes of the same (geometrical) kind or species. Thus, in this sense, lines are *neither* comparable nor incomparable to planes/surfaces, and planes/surfaces are neither comparable nor incomparable to solids—although there is an obvious intuitive sense (recognized by Greek geometers) in which magnitudes of different species are incomparable.

2. On the supposition that a real-valued function f of one variable defined for all x in the real open interval $(0, 1)$ is differentiable at a real number x_0 in this interval, the derivative of f at x_0 is the real number a (i.e., $f'(x_0) = (df/dx)_{x=x_0} = a$) just in case $\lim_{x \longrightarrow x_0}[(f(x) - f(x_0))/(x - x_0)] = a$. According to Weierstrass' 'ε, δ approach' this last condition is analyzed as follows: For any positive real ε there exists a positive real δ such that $|[(f(x) - f(x_0))/(x - x_0)] - a| < \varepsilon$ for all x in $(0, 1)$ such that $0 < |x - x_0| < \delta$.

3. Although there were other nineteenth-century constructions of the irrationals and reals, those of Cantor and Dedekind are no doubt the best known. See, for example, Kline [18], pp. 982–87.

4. This classical result may be obtained in a number of ways. For example, Pickert and Görke obtain it in terms of the isomorphic mappings of all complete (in the above sense: each nonempty set that is bounded above has a lub) ordered modules containing the rationals, where these mappings leave the rational numbers fixed and preserve order and addition (Pickert and Görke [23], pp. 134–46).

5. My thanks to an anonymous referee for *NDJFL* for impressing upon me the importance of this distinction.

6. The well-known argument is the following. Recollect that the compactness theorem states that a set Γ of wffs of a consistent first-order theory has a model (is satisfiable) just in case every *finite* $\Delta \subseteq \Gamma$ has a model. Consider a first-order fragment T of the theory of real numbers with a constant designating the binary irreflexive order relation $<$, a constant designating the set $\mathbb{N}$ of natural numbers, a 'canonical designation' for each standard natural number n (e.g., 0, 1, 2, 3, etc.), and a 'fresh' constant c. Let Γ be the (denumerably) infinite set $\Gamma = \{c \in \mathbb{N} \wedge c > 0, c \in \mathbb{N} \wedge c > 1, c \in \mathbb{N} \wedge c > 2, c \in \mathbb{N} \wedge c > 3, \ldots\}$. Since it is clear that there is a model of T that is also a model for all the finite subsets Δ of Γ (simply assign to c the successor of the largest natural number whose canonical designation occurs in Δ), the compactness theorem entails that there is a model for Γ as well. However, since no natural number is larger than itself, this model must assign to c a number larger than any of the standard numbers. Consequently, the model will not be Archimedean.

7. For alternative characterizations of the monomorphism *, see Hurd and Loeb [12], p. 79 and Robinson and Zakon [26], p. 111.

8. It is to be emphasized that this is a *semantic* condition. That is, the notation $\langle V(\mathrm{R}), \in\rangle \models \varphi[a_1, \ldots, a_n]$ means that a *sequence of objects* of the domain $V(\mathrm{R})$ of the model $\langle V(\mathrm{R}), \in\rangle$ satisfies the bounded-quantifier formula $\varphi(x_1, \ldots, x_n)$ with respect to that model $\langle V(\mathrm{R}), \in\rangle$. Similarly, $\langle V({}^*\mathrm{R}), \in\rangle \models \varphi[{}^*a_1, \ldots, {}^*a_n]$ means that the sequence of objects $\langle {}^*a_1, \ldots, {}^*a_n\rangle$ of the domain $V({}^*\mathrm{R})$ of the model $\langle V({}^*\mathrm{R}), \in\rangle$ (which is the *-image of the sequence $\langle a_1, \ldots, a_n\rangle$) satisfies the *same* bounded-quantifier formula $\varphi(x_1, \ldots, x_n)$ with respect to $\langle V({}^*\mathrm{R}), \in\rangle$

9. Another way of characterizing the saturation of an elargement over $V(\mathrm{R})$ begins with the idea of a *concurrent* binary relation $S \in V(\mathrm{R})$: a relation such that for any *finite* number of elements $a_1, \ldots, a_m$ of its domain, there is some $b \in V(\mathrm{R})$ such that $(a_i, b) \in S$, for each $i = 1 \ldots m$. An enlargement of the superstructure $V(\mathrm{R})$ (or nonstandard universe $\langle V(\mathrm{R}), V({}^*\mathrm{R}), {}^*\rangle$ that is saturated over $V(\mathrm{R})$), then, is one for which, for each concurrent relation $S \in V(\mathrm{R})$, there is some $c \in V({}^*\mathrm{R})$ such that $({}^*x, c) \in {}^*S$ for all x in the domain of relation S simultaneously. The monomorphism * is then said to *bound* all concurrent relations; and such 'added' elements $c \in V({}^*\mathrm{R})$ are sometimes said to be *ideal*. There are also other more general model-theoretic characterizations of saturation (and of α-saturation for a cardinal α) that I shall not discuss in this essay.

10. A striking elementary example is the very succinct and elegant nonstandard proof (unfortunately marred by some typographical errors) by Robinson of the (standard) intermediate value theorem for continuous functions ([25], p. 67, Theorem 3.4.6).

11. An anonymous commentator on this paper reports that, while Nelson [22] was published a few months previously, Hrbáček [11] had been submitted prior to the submission of Nelson's paper. The same commentator notes that the two theories were developed completely independently.

12. A galaxy is a set (equivalence class) of hyperreal numbers such that any two members of the set are a finite distance (in the standard sense of that phrase) apart.

13. An anonymous commentator on the present paper represents a different reaction: "One might retort that this is so [i.e., that IST is not Janus-faced in the way that most developments of nonstandard mathematics are], not for any intrinsic reasons, but simply because IST chooses to wear blinders on its backward-looking face."

14. Several of these later axiomatic developments introduce two 'new' unary predicates, 'is standard' and 'is internal'. Hrbáček weakens Nelson's idealization (saturation) axiom, strengthens Nelson's standardization axiom, and provides for external sets satisfying a limited number of ZFC axioms. It was obvious that a well-foundedness axiom cannot apply to external sets. Hrbáček showed that neither of the combinations of replacement and choice axioms or replacement and power set axioms can apply to external sets (Hrbáček [11]). Kawai has constructed an axiomatic nonstandard set theory much like Hrbáček's but with a stronger idealization and considerably weaker standardization axiom, which allows all of the ZFC axioms except well-foundedness to apply to external sets (Kawai [14]). Fletcher has developed a "stratified nonstandard set theory" (SNST) along similar lines, but with an idealization schema that is intended "to allow only as much idealisation as is needed" ([10], p. 1005). Ballard—in his monograph containing a very useful synoptic account of the set-theoretic development of the foundations of nonstandard mathematics—has proposed an "enlargement set theory" (EST), which is a more radical approach than Fletcher's to variable idealization and, according to which, "for each cardinal κ, the internal universe I_κ shall be a κ-saturated elementary extension, not of S [the universe of standard sets], but of E_κ [the variable universe of external sets]" ([1], p. 101). Finally, in a series of three articles, Kanovei and Reeken have developed a variant of IST, bounded set theory (BST), which is a theory for those IST sets that are members of standard sets. An enlargement of the BST universe satisfies the axioms of HST, "an external theory close to a theory introduced by Hrbáček" (Kanovei and Reeken [13]).

15. A *strong limit cardinal* is a cardinal α such that for any cardinal β less than (included in) α, it is also the case that $2^{\beta} < \alpha$. Where β is a limit ordinal, the *cofinality of* β (cf(β)) is the least cardinal α such that there is a set X of that cardinality for which both $X \subset \beta$ and $\bigcup X = \beta$. A cardinal α is regular just in case cf(α)= α. Finally, a strong limit cardinal that is regular is *inaccessible*. $\omega(= \aleph_0)$ is an inaccessible cardinal. An *uncountable inaccessible* cardinal would be a larger inaccessible cardinal. The assumption of the nonexistence of uncountable inaccessible cardinals is consistent with Zermelo-Fraenkel set theory. The *axiom of inaccessible sets* (due to Tarski), providing for the existence of such uncountable inaccessible cardinals is, in the words of Chang and Keisler, "like the [Generalized Continuum Hypothesis] . . . a plausible extra assumption about set theory that can usually be avoided" ([3], p. 590). But not avoided, it seems, for obtaining this 'uniqueness' theorem. See the following discussion in the text.

16. I am grateful to an anonymous commentator on this paper for suggesting this idea to me.

17. While $^{\sigma}C$ is usually considered to be a set of $V(^*\mathrm{R})$, in some notations '^{0}C' designates a set 'of classical real analysis'—that is, not a set of $V(^*\mathrm{R})$, but of the original (standard) model $V(\mathrm{R})$: where C is a set of $V(^*\mathrm{R})$, $^{0}C = \{x : ^*x \in C\} \in V(\mathrm{R})$.

REFERENCES

[1] Ballard, D., *Foundational Aspects of "Non"standard Mathematics*, American Mathematical Society, Providence, 1991.

[2] Bostock, D., *Logic and Arithmetic, Rational and Irrational Numbers*, vol. 2, Clarendon Press, Oxford, 1979.

[3] Chang, C. C., and H. J. Keisler, *Model Theory*, 3d edition, North-Holland, Amsterdam, 1990.

[4] Conway, J. H., *On Numbers and Games*, Academic Press, London, 1976.

[5] Dauben, J. W., *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Princeton University Press, Princeton, 1979.

[6] Ehrlich, P., "All numbers great and small," pp. 239–58 in *Real Numbers, Generalizations of the Reals, and Theories of Continua*, edited by P. Ehrlich, Kluwer Academic Publishers, Dordrecht, 1994.

[7] Ehrlich, P., "General introduction," pp. vi–xxxii in *Real Numbers, Generalizatons of the Reals, and Theories of Continua*, edited by P. Ehrlich, Kluwer Academic Publishers, Dordrecht, 1994.

[8] Ehrlich, P., "Hahn's *Über die nichtarchimedishen Grössensysteme* and the development of the modern theory of magnitudes and numbers to measure them," pp. 165–213 in *From Dedekind to Gödel: Essays on the Development of the Foundations of Mathematics*, edited by J. Hintikka, Kluwer Academic Publishers, Dordrecht, 1995.

[9] Feferman, S., "Does mathematics need new axioms?," *The American Mathematical Monthly*, vol. 106 (1999), pp. 99–111.

[10] Fletcher, P., "Nonstandard set theory," *The Journal of Symbolic Logic*, vol. 54 (1989), pp. 1000–08.

[11] Hrbáček, K., "Axiomatic foundations for nonstandard mathematics," *Fundamenta Mathematicae*, vol. 98 (1978), pp. 1–19.

[12] Hurd, A. E., and P. A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, Orlando, 1985.

[13] Kanovei, V., and M. Reeken, "Internal approach to external sets and universes: Part 1: Bounded set theory," *Studia Logica*, vol. 55 (1995), pp. 229–57; "Part 2: External universes over the universe of bounded set theory," *Studia Logica*, vol. 55 (1995), pp. 347–76; "Part 3: Partially saturated universes," *Studia Logica*, vol. 56 (1996), pp. 293–322.

[14] Kawai, T., "Nonstandard analysis by axiomatic method," pp. 55–76 in *Southeast Asian Conference on Logic: Proceedings of the Logic Conference*, edited by C. -T. Chong and M. J. Wicks, North-Holland, Amsterdam, 1983.

[15] Keisler, H. J., *Elementary Calculus*, Prindle, Weber, and Schmidt, Inc., Boston, 1976.

[16] Keisler, H. J., *Foundations of Infinitesimal Calculus*, Prindle, Weber, and Schmidt, Inc., Boston, 1976.

[17] Keisler, H. J., "The hyperreal line," pp. 207–37 of *Real Numbers, Generalizations of the Reals, and Theories of Continua*, edited by P. Ehrlich, Kluwer Academic Publishers, Dordrecht, 1994.

[18] Kline, M., *Mathematical Thought from Ancient to Modern Times*, vol. 3, Oxford University Press, Oxford, 1972.

[19] Lear, J., "A note on Zeno's arrow," *Phronesis*, vol. 26 (1981), pp. 91–104.

[20] McLaughlin, W. I., "Thomson's lamp is dysfunctional," *Synthese*, vol. 116 (1998), pp. 281–301.

[21] McLaughlin, W. I., and S. L. Miller, "An epistemological use of nonstandard analysis to answer Zeno's objections against motion," *Synthese*, vol. 92 (1992), pp. 371–84.

[22] Nelson, E., "Internal set theory: a new approach to nonstandard analysis," *Bulletin of the American Mathematical Society*, vol. 83 (1977), pp. 1165–98.

[23] Pickert, G., and L. Görke, "Construction of the system of real numbers," pp. 93–165 in *Fundamentals of Mathematics*, vol. 1: *Foundations of Mathematics; The Real Number System and Algebra*, edited by H. Beneke et al., The MIT Press, Cambridge, 1974.

[24] Robert, A., *Nonstandard Analysis*, translated by A. Robert, John Wiley and Sons, New York, 1988.

[25] Robinson, A., *Nonstandard Analysis*, revised edition, Princeton University Press, Princeton, 1996.

[26] Robinson, A., and E. Zakon, "A set-theoretical characterization of enlargements," pp. 109–22 in *Applications of Model Theory to Algebra, Analysis, and Probability*, edited by W. A. J. Luxemburg, Holt, Rinehart, and Winston, New York, 1969.

[27] Sinaceur, H., "Calculation, order and continuity," pp. 191–206 in *Real Numbers, Generalizations of the Reals, and Theories of Continua*, edited by P. Ehrlich, Kluwer Academic Publishers, Dordrecht, 1994.

[28] Stroyan, K. D., and W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York, 1976.

[29] White, M. J., "Zeno's arrow, divisible infinitesimals, and Chrysippus," *Phronesis*, vol. 26 (1982), pp. 239–54.