Presentation for History of Logic Colloquium–University of Cincinnati

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We are now quite accustomed to the view that the extension of the field of rational numbers to the field of real numbers is a natural, useful, and proper move–irrespective of the precise mathematical methods that we may prefer in order to effect that extension and irrespective of our ontological persuasions (or lack thereof) with respect to (rational and real) numbers and other mathematical objects. To someone of a philosophical cast of mind contemplating the development of nonstandard analysis over the last forty years, it might well be assumed that, by parity of whatever reasoning that could be invoked to justify this view concerning the relation of the rationals to the reals, we should similarly be prepared to regard the extension of the field of real numbers to the field of hyperreal numbers as a natural, useful, and proper move. It is my aim in this essay to explore this assumption first by delving into some of the mathematical 'archaeology' or history of incommensurables (magnitudes that do not have a common aliquot part) and of what I call incomparables (magnitudes such that the larger can never be surpassed by any finite number of additions of the smaller to itself). What I shall suggest is that the contemporary development of the nonstandard or non-Archimedean mathematics of hyperreal numbers has a certain 'Janus-faced' character that militates against a paradigm-shift that would result in the displacement of the real by the hyperreal number line. A corollary that I'll be particularly emphasizing today is that this Janus-faced character renders the use of nonstandard or non-Archimedean concepts problematic as tools for addressing certain classical conceptual–or, if you will, philosophical–problems pertaining to continuous magnitudes.

I. The Ideas of Incommensurability and Incomparabililty.

According to canonical lore, it was those mythical, mystical Pythagoreans who discovered that there are some magnitudes that are incommensurable with others. In addition to constituting a specifically mathematical conundrum, this discovery supposedly was regarded as generally very bad news by the Pythagoreans, who wished to develop a numerical ontology of everything. If it is not *even* the case that there is any length of which both the side and diagonal of the square are integral multiples, it would seem to be difficult for the Pythagoreans to make a convincing case for their claim (reported by Aristotle) that the "elements of numbers are the elements of all things that exist" (*Meta* .1.5.986a1-2) and, more particularly, that "justice is a certain property of numbers, soul and reason another, and opportunity still another, and similarly for each other thing" (Ibid., 985b29-31). In fact, the discovery of incommensurability was regarded as so disastrous, according to the hoary tradition to which I am appealing, that a massive Pythagorean cover-up was undertaken. An apostate Pythagorean whistle-blower, Hippasus of Metapontum, was punished for spreading the news beyond the confines of the Pythagorean brotherhood, according to alternative accounts, either directly by the Pythagoreans (who erected a tombstone to him despite the fact that he was not yet dead) or more sharply and by higher powers (who brought about his death at sea by shipwreck).

In contemporary (and quite un-Greek) terminology, we see that the rational numbers are not closed under the operation of taking the square root: √2 (length of the diagonal of a square with unit side) is not expressible as a ratio of integers. In this sense–a sense that obviously has implications even for elementary plane geometry–the rational numbers are 'incomplete'. Would it not make good sense to 'fill in the gaps' by adding to the rationals some 'new' numbers by which we can represent the length of the diagonal of the unit square, as well as other constant, incommensurable (and, in fact, 'transcendental') ratios such as that of the circumference of a circle to its diameter.

Sensible or not, it was a move that Greek geometers did not make. Rather, the Greek response was to draw a sharp distinction between discrete πλῆθος (multiplicity or plurality) and continuous μέγαθος (magnitude) and to locate the phenomenon of incommensurability squarely in the latter category. Distinguishing these two coordinate kinds of quantity (πόσον), Aristotle characterizes multiplicity as numerable quantity and magnitude as measurable quantity (*Meta*. 5.13.1020a8-10), maintaining that "number (ἀριθμός) is commensurable (σύμμετρος), but number is not predicated with respect to what is not commensurable" (*Meta*. 5.15.1021a5). And, as we know, the 'theory of incommensurability' is explicated by Euclid in his theory of proportionality *of magnitudes* in the fifth and tenth books of the *Elements*. The distinction in Greek mathematics between multiplicities and magnitudes at least partly explains the parallel development of a theory of proportionality *for numbers* in the seventh book of the *Elements*.

Although, of course, numerical representation of and algebraic application of incommensurable magnitudes long antedated the nineteenth century, such mathematical practice received its full theoretical justification only with the work of nineteenth-century mathematicians such as Cauchy, Dedekind, and Cantor. It is this work that allows the ordered field of reals to be seen as the 'completion' of the ordered field of rationals. This arithmeticization of continuous magnitudes constituted the final–or, at least, final 19th century–solution to the Greek conundrum of incommensurable magnitudes.

The second geometrical phenomenon, which I referred to as that of incomparability, arises in Greek attempts to ascertain, say, the area of circle or the area of a parabolic segment in terms of the limit of areas of inscribed polygons, or the volume of a cylinder in terms of the limit of the areas of inscribed prisms. Would it make good sense to introduce the concept of positive but really little–and I mean *really* little–magnitudes so that, say, there is only a vanishingly small difference between the volume of a cylinder and the volume of an inscribed prism with a sufficiently large number of faces. Such a really little magnitude would be 'vanishingly small', 'insignificant', or 'unassignable' relative to any 'standard' magnitude in the following sense. It would not 'have a ratio' (λόγον ἔχειν) to the larger standard magnitude in the terms of Definition 4 of the fifth book of the *Elements*: "a magnitude is said to have a ratio to another when it can, when multiplied, exceed the other." As previously indicated, I use the term 'incomparable' to designate such magnitudes that would not (as we now say) stand in any real-valued ratio to one another.

Again, whether sensible or not, the introduction of quantities that are incomparable in this sense seems not to have been a move made within the mainstream of Greek mathematics. In fact, Archimedes is known for having stated in the first book of *De sphaera et cylindro* a postulate (the Archimedean axiom) that requires that the difference between any two unequal magnitudes of the same kind (i.e., lines, surfaces, solids) and any other magnitude of that kind 'have a ratio to one another' in the sense of Def. 4 of the fifth book of Euclid. Euclid himself had, in effect, used his definition as a postulate (i.e., had applied it to any unequal magnitudes) in the proof of Prop. 1 of the tenth book of the *Elements*, known as the Euclidean lemma: for magnitudes x, y such that $x > y$, the subtraction from x of a magnitude greater than half of x, the subtraction from this remainder *r* of a magnitude greater than half of *r*, etc. will eventually (i.e., in a finite number of steps) yield a remainder *z* less than *y*.

So, adoption of the Archimedean axiom rules out the possibility of incomparable magnitudes: it rules out magnitudes that would be infinitesimal relative to standard, finite quantities as well as infinitely large quantities, which would be larger than any standard finite quantities and equal to the multiplicative inverses of infinitesimals. The combination, then, of the Archimedean axiom and *reductio* proof provides a means of utilizing an informal concept limit in a finitistic context (*viz.*, the availability of *finite* differences between variable quantities *V* and some fixed quantity *L*, that can be made 'as small as one wishes') without actually 'passing to the limit'–i.e., without, that is, actually ever identifying the volume of a cylinder with that of some inscribed prism (perhaps one with an infinite number of faces) in the sense of claiming that their magnitudes differ by an amount that is incomparable–in my sense of the term–with the volumes of either cylinder or inscribed prism. This perspective became the mathematically orthodox one–now unfortunately designated, as most of you are no doubt aware, by the phrase "method of exhaustion. [Say something about 'method of indivisibles' as a heuristic 'method of discovery': Archimedes' *Methodus* (῞Εφοδος) and Cavalieri's work in the 17th century.]

To conclude these brief remarks concerning incommensurables and incomparables in Greek mathematics, I wish to emphasize the point that classical Greek mathematics found perfectly comfortable (if, at times, a bit complicated and unwieldy) geometrical means for dealing with both phenomena: the theory of proportion for the analysis of incommensurables; a combination of the method of exhaustion, as a method of proof, and (at least in some cases) the method of indivisibles, as a method of discovery, for infinitary analysis. What is completely lacking is any attempt to accommodate the two phenomena *numerically*. Number, for the Greeks, is a concept that remains limited to what we would call *natural numbers*.

II. An Illicit–but Brief–Excursus into 19th Century Mathematics.

According to contemporary mathematical orthodoxy, a satisfactory and final resolution of both the ancient conundrums that we have been considering–that pertaining to incomparables (and, more generally, the employment of infinitary analytical techniques) and that pertaining to incommensurables–was achieved in the nineteenth century. Although he was in certain respects anticipated by Cauchy and Bolzano, Karl Weierstrass is usually credited with fully appreciating that the limit concept could be developed, in terms of the so-called 'ε, δ approach', in such a way as to eliminate any temptation to appeal to an idea of infinitesimal or infinitely small (and infinitely great) quantities. Talk of 'infinitesimal differences' or differentials increasingly came to be regarded as a mere *façon d'parler*. [I won't go into details.] The nineteenth-century fate of incommensurables was, in a certain respect, diametrically opposed to the nineteenth-century fate of incomparables: with the arithmeticization of the continuum by Dedekind and Cantor, relatively incommensurable magnitudes found a secure *numerical* home as irrationals. Dedekind and Cantor each developed techniques, now regarded as standard, for the 'construction' of the real numbers from the rationals. [Again, I won't go into details: Dedekind cuts and Cantor's similar construction using Cauchy sequences.]

In other words, as a result of nineteenth-century developments with respect to the two classical Greek phenomena of incommensurable and incomparable magnitudes, the former was arithmeticized–incorporated into a system of numbers–while the latter was not. And, as Philip Ehrlich has pointed out, a consequence of the so-called Cantor-Dedekind axiom identifying the arithmetical continuum with the geometrical continuum is that incomparable magnitudes are finally and firmly denied even any *geometrical* existence.

The Cantor-Dedekind perspective is perhaps given added support by a classical uniqueness result (originating, I believe, with David Hilbert and later clearly stated by Hans Hahn) for the arithmetic continuum or ordered field of real numbers:

> (Real Completeness Theorem): The ordered field $\mathbb{R}$ of real numbers is, up to order isomorphism, the one ordered field that is *complete* in the following sense: any nonempty subset that has an upper bound in $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.

It might thus seem that any algebraic proper extension of the field $\mathbb{R}$ of real numbers must sacrifice in some way the 'completeness' of the reals. And, in a sense, this is true: any such extension must sacrifice the Archimedean property (and its logical equivalents) of the ordered field. But we already knew that the addition of incomparable magnitudes (infinitesimals and infinitely large numbers, relative to the standard reals) to the ordered field of reals would entail sacrifice of the Archimedean property. The question is whether there are *other* significant senses of 'completion' in which an ordered field containing representations of incomparable magnitudes might be regarded as a completion of $\mathbb{R}$.

As the Cantor-Dedekind perspective gradually achieved the status of mathematical orthodoxy, non-Archimedean perspectives became–to use a currently fashionable term–increasingly marginalized. The attitude of Cantor himself seems to have played a part in this process. His own opposition to infinitesimals was unswerving and, at times, vitriolic. In a letter to Vivanti, he credited Thomae with being the first to "infect mathematics with the Cholera-Bacillus of infinitesimals"; and he suggests that, in developing the ideas of Thomae, du Bois-Reymond found "excellent nourishment for the satisfaction of his own burning ambition and conceit." In effect, Cantor denied the possibility of infinitesimal numbers because he believed that the Archimedean principle, or something equivalent to it, was entailed by the concept of (linear) number. Therefore, by the so-called Cantor-Dedekind axiom, there is no legitimate geometrical conception of an infinitesimal magnitude either. In the words of Joseph Dauben,

> had Cantor agreed that the Archimedean property of the real numbers was merely axiomatic, then there was no reason to prevent the development of number systems by merely denying the axiom, so long as consistency was still preserved. But to have allowed this would have left Cantor open to the challenge that, if infinitesimals could be produced without contradiction, then his own view of the continuum was lacking and the completeness of his own theory of number would have been contravened.

Of course, as has been previously noted, no non-Archimedean ordered field is complete in the mathematical sense specified above--or in any intuitive sense that is equivalent to or entailed by the Archimedean axiom. However, an ordered field is characterized not just by the order relation but by the algebraic field operations; and if we shift our attention from the order relation to these algebraic operations, a rather different intuitive picture of 'completeness' emerges.

> An ordered field $\mathbb{F}$ is *real closed* just in case both (i) every positive element of $\mathbb{F}$ has a square root in $\mathbb{F}$ and (ii) every polynomial of odd degree has a root in $\mathbb{F}$.

Artin and Schreier investigated the theory of real closed fields in the 1920s showing that the ordered field $\mathbb{R}$ of real numbers is a real closed field. The ordered field $\mathbb{Q}$ of rationals clearly is not: there is no element of $\mathbb{Q}$, for example, that is the square root of the element 2. There is a fairly intuitive sense in which the concept of a real closed field represents a sort of 'algebraic completeness'. But in turns out that this concept of completeness is *not* uniquely satisfied by the reals up to isomorphism. There are other proper extensions of the reals that, while non-Archimdean, are real closed. The non-Archimedean character of such extensions means, in effect, that they add 'real infinitesimals (and their inverses) to the real line. One such extension is the reall closed field of hyperreals constructed using model-theoretic techniques by Abraham Robinson in the lat 1950s and early 1960s. We thus perhaps have the option of overturning 19th century mathematical orthodoxy and making a *numerical* home for incomparable magnitudes as well as incommensurable ones.

III. The Model-Theoretic 'Superstructure' Approach to the Construction of the Hyperreals.

The superstructure approach works with languages with a constant designating the binary set-membership relation $\in$ and bounded quantifiers defined in terms of it $[(\forall x \in y), (\exists x \in y)]$. We shall let the base set R of a superstructure V(R) be the set of real numbers. The nth cumulative power set is defined inductively as V_0(R) = R; V_{n+1}(R) = V_n(R) $\cup$ $\wp$(V_n(R)). Then the superstructure V(R) over R is the union of all the cumulative power sets V_i. In view of the standard method for defining ordered sequences in terms of sets, n-ary functions as n + 1-ary relations, etc. it is fairly clear that a superstructure contains the properties of reals, relations on reals, functions and operations on reals, "function spaces, measures, and all other structures from classical analysis." The 'trick' of the method of superstructures is to find *another* superstructure V(*R) and a *monomorphism* or injective mapping * from V(R) into V(*R) satisfying certain conditions. The result is an ordered triple ⟨V(R), V(*R), *⟩ that Chang and Keisler call a *nonstandard universe*. Let ⟨V(R), $\in$⟩ be a model with $\in$ the binary set-membership relation. The essential conditions that ⟨V(R), V(*R), *⟩ must satisfy in order to be a nonstandard universe are (in addition to the requirement that R and * R be infinite, which we have already assumed):

(i) that (as the notation already suggests) * maps the base set R of V(R) to the base set *R of V(*R);

(ii) that the mapping * be a bounded elementary embedding of $V(\mathrm{R})$ into $V(^*\mathrm{R})$–that is, for every bounded-quantifier well-formed formula ϕ with n free variables and every $r_1, \ldots, r_n \in \mathrm{R}$, $r_1, \ldots, r_n$ satisfy ϕ in $\langle V(\mathrm{R}), \in\rangle$ (i.e., $\langle V(\mathrm{R}), \in\rangle \models \phi[r_1, \ldots, r_n]$) if and only if $^*r_1, \ldots, {}^*r_n$ satisfy ϕ in $\langle V(^*\mathrm{R}), \in\rangle$ (i.e., $\langle V(^*\mathrm{R}), \in\rangle \models \phi[^*r_1, \ldots, {}^*r_n]$).

(iii) that, for every infinite $\mathrm{X} \subseteq \mathrm{R}$, $^o\mathrm{X} = \{^*r: r \in \mathrm{X}\} \subsetneq {}^*\mathrm{X}$.

We also desire that the nonstandard universe $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ be an *enlargement* or *saturated over* $V(\mathrm{R})$. There are a number of ways of characterizing saturation in such a case. One way is the following. Suppose that we extend our bounded-quantifier language for the reals (with a constant '$\in$' for set membership) by adding a constant for each element of $V(\mathrm{R})$. Then

$\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ is saturated over $V(\mathrm{R})$ if and only if, for every $n < \omega$ and for every set Γ of bounded-quantifier formulas of this expanded language, *if* every *finite* $\Delta \subseteq \Gamma$ is satisfiable in $\langle V(\mathrm{R}), \in\rangle$ by elements of $V_n(\mathrm{R})$, then Γ is satisfiable in $\langle V(^*\mathrm{R}), \in\rangle$ by an element of $V_n(^*\mathrm{R})$.

Another way of characterizing an enlargement/saturation begins with the idea of a *concurrent* binary relation $\mathrm{S} \in V(\mathrm{R})$: a relation such that for any *finite* number of elements $a_1, \ldots a_m$ of its domain, there is some $b \in V(\mathrm{R})$ such that $(a_i, b) \in \mathrm{S}$, for each $i = 1 \ldots m$. An enlargement of the superstructure $V(\mathrm{R})$ (or nonstandard universe $\langle V(\mathrm{R}), V(^*\mathrm{R}), ^*\rangle$ that is saturated over $V(\mathrm{R})$), then, is one for which, for each concurrent relation $\mathrm{S} \in V(\mathrm{R})$, there is some $c \in V(^*\mathrm{R})$ such that $(^*x, c) \in {}^*\mathrm{S}$ for all x in the domain of relation S simultaneously. The monomorphism * is then said to *bound* all concurrent relations; and such 'added' elements $c \in V(^*\mathrm{R})$ are sometimes said to be *ideal*. [There are also other more general model-theoretic characterizations of saturation (and of α-saturation for cardinal α) that I shall not discuss here.]

This construction nicely suggests how nonstandard universes introduce *numerical* elements (members of non-Archimedean ordered fields) to represent quantities that were once regarded as 'ideal' in something like the sense of Leibniz: mathematical fictions introduced to shorten the process of reasoning. For example, saturation yields the result that, since there is some $N > n$ for each n that is a member of any *finite* subset of the set $\mathbb{N}$ of natural numbers, there is some 'ideal' element $c \in {}^*\mathbb{N}$ such that $c \;^*\!\!> n$, for every $n \in \mathbb{N}$. It can then be shown that there is another 'ideal' numerical element $c^{-1} = 1/c \in V(^*\mathrm{R})$, the multiplicative inverse of c, that is infinitesimal, non-zero and nonnegative but less than (*<) any standard hyperreal ('embedded real') $^*r$ such that $r \in$ R.

Along with the saturation or enlargement characteristic of nonstandard universes, the other crucial characteristic of nonstandard universes is that the monomorphism * of such universes be characterized by what is usually referred to as the *transfer principle* (or *Leibniz' principle*). Starting with the requirement (ii) above that the injective mapping * be a bounded elementary embedding of V(R) into V(*R), we obtain a way of taking any sentence 'of the language of the reals' and its intended interpretation in V(R) and constructing, with the use of the monomorphism *, a hyperreal interpretation of it (its '*-transform') in V(*R). The resulting *biconditional* is the transfer principle or Leibniz' principle: a bounded well-formed formula (wff) ϕ is true if and only if (iff) its *-transform (signified by '$^*\phi$') is true. Consequently, the hyperreal superstructure V(*R) of a nonstandard universe–or of the 'theory' of the hyperreals–is a *conservative extension* of the real superstructure V(R)–or of the 'theory' of the reals. That is, any true claim about the reals has a corresponding *-transform true claim about the hyperreals; and any true claim about the hyperreals *that can be represented as the *-transform of a claim about the reals* has a corresponding true claim (that claim of which it is the *-transform) about the reals. David Ballard aptly describes such a transfer principle as insuring that a nonstandard theory "is 'safe' for conventional mathematicians." In particular, the transfer principle supplies a new, and sometimes useful (or elegant, or interesting, or relatively more simple), way of proving theorems about the real numbers: viz., the 'nonstandard' proof of a claim about hyperreal numbers that can be represented as the *-transform of a claim about the reals.

The *-transforms of first-order propositions about the reals (propositions that involve quantification over only the real numbers themselves) have essentially the same 'sense' as the first-order propositions about the reals of which they are the *-transforms. Thus, for example, corresponding to the claim that, for every positive real number (element of R), there is a smaller positive real number, there is a straightforward analogous claim (*-transform) about positive hyperreal numbers (elements of *R). However, while higher-order propositions about the reals have true *-transforms, they exhibit a more marked 'shift in meaning'. To consider a concrete example, the ordered field of reals is 'complete' in the sense defined above: every nonempty subset $X \subseteq R$ that has an upper bound has a least upper bound (supremum). This characteristic can be formulated as a second-order truth about the reals–second-order because it involves quantification over sets of reals or, in terms of our restriction to bounded quantifiers, over elements of the power set of the reals, $\wp(R)$. We face here what initially appears to be an antinomy. There is a true *-transform, i.e., a 'corresponding' truth about the hyperreals, to the second-order proposition expressing the completeness of the ordered field of reals. But, because the ordered field of hyperreals in non-Archimedean, there will be sets of hyperreals (elements of $\wp(^*R)$) which are bounded above but have no *least* upper bound (e.g., among many other sets, the set of positive infinitesimal hyperreals). The antinomy is only apparent because the bounded second-order quantifier in the *-transform of the completeness proposition rangers over the *-transform of the power set of the reals, i.e., $^*\wp(R)$, and this set is not identical to (is, in fact, a *proper* subset of) the power set of the hyperreals, i.e., $\wp(^*R)$.

To speak informally, there are entities of the hyperreal superstructure $V(^*R)$–here, certain sets of hyperreals or elements of $\wp(^*R)$–that the *-transforms of truths about the reals fail 'to detect' or 'to know about'. These are just the entities that would 'make trouble'–in our particular example, sets of hyperreals or elements of $\wp(^*R)$ that, while bounded above, do not have a least upper bound. This situation is the manifestation of an important distinction among three kinds of entities of the hyperreal superstructure $V(^*R)$. A *standard* entity y is an element of $V(^*R)$ that is the image, in terms of the monomorphism *, of some x of $V(R)$ [i.e., $(\exists x \in V(R))(y = {}^*x)$]. An *internal* entity y is an element of $V(^*R)$ that is an element of some standard entity x that is itself an element of $V(^*R)$ [i.e., $(\exists z \in V(R))(\exists x \in V(^*R))(x = {}^*z \wedge y \in x)$]. It is easy to show that all standard entities are internal ones, but the converse does not hold. Finally, an *external* entity is an element of $V(^*R)$ that is not internal.

Since it is internal entities that bounded quantifiers range over, it is these entities that our theory of reals, when interpreted in the nonstandard model (the superstructure $V(^*R)$) 'detects' or 'knows about'. The theory of reals so interpreted does *not* detect/know about external entities, which thus have a sort of 'ghostly' presence in the model. I shall later return to the philosophical implications of this ghostly presence. For the moment, I merely note some straightforward but perhaps somewhat surprising consequences of the distinction among standard, internal, and external entities. The *-image *r of each and every real number r (member of R) is a standard entity; and, indeed, such images are customarily thought of as simply being the real numbers 'embedded' into the set *R of hyperreals. However, *R contains infinitesimal and infinitely large elements that are not the *-images of any real numbers; hence such infinitesimal and infinitely large hyperreals are not standard but *are* internal (because they are members of the standard entity *R). However, the *set* $^{\sigma}$R = {*r ∈ *R: r ∈ R} of all and only the 'embedded reals' is an external set–as is the set of all and only infinitesimal elements of *R and the set of all and only infinitely large elements of *R. Consequently, we can say that, although the theory of reals, when interpreted in the nonstandard model V(*R), detects or knows about the 'embedded' or standard reals 'individually' or 'distributively', it has 'lost sight' of the *set* of standard reals. In fact, it turns out that any *infinite* set containing all and only standard reals is external. One might say that, with respect to the interpretation of the theory of reals within the nonstandard model V(*R), the standard reals become 'indistinguishable' from nonstandard hyperreals in infinite sets.

IV. Some Philosophical Implications.

After the technicalities of the preceding section, I return to the principal conceptual question of this talk: Is the nonstandard method of according numerical status to incomparables –by means of the hyperreal extension of the field of real numbers–a simple and straightforward analogue of the nineteenth-century method of according numerical status to incommensurables–by means of the real extension of the field of rational numbers? As I have now indicated, there certainly are some similarities, beginning with an informal picture of the number line. The 'construction' of the reals by Dedekind and Cantor suggests the filling in of 'gaps' in the rational line by irrational numbers. Similarly, according to a common picture of the hyperreals, infinitesimals 'fill in the gaps' surrounding each standard, finite real (yielding monads); and the negative and positive directions of the real line are extended by 'galaxies' of hyperfinite reals. In his text *Elementary Calculus*, which is based on Robinson's development of nonstandard analysis, Keisler uses the heuristic devices of the 'infinitesimal microscope' and the 'infinite telescope' for 'looking at'–for example–the behavior of the slope of a 1-place function within the 'monadic neighborhood' of a given ordered pair of points. [In fact, these devices are formally defined in

Keisler's *Foundations of Infinitesimal Calculus*.]

We saw that the field ℝ of reals numbers is the unique *complete* ordered field up to isomorphism. Since the non-Archimedean field *ℝ of hyperreals is not isomorphic to the reals, it is, of course, not complete in the technical sense that we earlier noted: every nonempty subset X ⊆ R that is bounded above has a least upper bound. For example, the (external) set of infinitesimals, while bounded above (by any standard real), has no supremum: there is no greatest infinitesimal nor smallest standard real number. An analogous uniqueness theorem for the hyperreal number system is more difficult to construct because of the fact that nonstandard models of the theory of real numbers–such as the hyperreal nonstandard universes ⟨*V*(R), *V*(*R), *⟩–of arbitrarily large cardinality can be found. However, it turns out that there is such a uniqueness theorem: If we require (i) that the monomorphism * of a nonstandard universe ⟨*V*(R), *V*(*R), *⟩ satisfies the transfer principle, (ii) that ⟨*V*(R), *V*(*R), *⟩ be saturated over *V*(R), and (iii) that both *R and the set of all internal sets have the cardinality of the first uncountable inaccessible cardinal, there is, up to isomorphism, a unique such nonstandard universe ⟨*V*(R), *V*(*R), *⟩.

Despite such similarities, there is a fundamental difference between the classical nineteenth-century approach to incommensurables and the twentieth-century approach to incomparables. As the term 'nonstandard' suggests, nonstandard or non-Archimedean hyperreal models were developed as alternative, nonstandard models of the *theory of real numbers*. A principal motivation for the development of nonstandard analysis was, in the words of Robinson, the conviction "that the theory of certain types of non-archimedean fields can indeed make a positive contribution to classical Analysis."

So it is perhaps not surprising that most developments of non-Archimedean mathematics have had the Janus-faced character to which I earlier alluded: they look *both* 'back' to the standard, intended interpretation or model of (a part of) the mathematical theory of the real number system *and* 'forward' to a nonstandard (non-Archimedean) model of that same theory. This Janus-faced character is made particularly perspicuous by the formal representation ⟨*V*(R), *V*(*R), *⟩ of a nonstandard universe, which contains both a superstructure on the reals and a superstructure on the hyperreals, together with the monomorphism * connecting them. It is clear that, at least with respect to infinite sets, the two superstructure models are incompatible in the sense of assigning infinite sets that are not isomorphic as extensions of the relevant predicates–e.g., ℕ and *ℕ to 'is a natural number, *R and R to 'is a real number', similarly for 'is finite', etc. As we saw, *within the nonstandard model*, the former infinite sets of standard elements possess only a ghostly existence as external sets, which the bounded quantifiers do not recognize. A ghostly but important existence, however. If the principal value of such non-Archimedean models is to serve as one element, among many, in the tool kit of mathematicians working in classical analysis or some other area of Archimedean mathematics, it is crucial not to let any of the entities of the original intended Archimedean model–or other important external sets of the nonstandard model–disappear.

The situation with respect to the nineteenth-century construction of the reals from the rationals was quite different. It certainly was never supposed that the (non-Archimedean) ordered field ℚ of rationals and the (Archimedean) field ℝ of reals were alternative–and in a sense incompatible--models or interpretations of the *same* mathematical theory. The added irrational elements of the latter were simply numerical representations of entities that had possessed, since at least Greek antiquity, a well-established and respectable geometrical presence–whereas the numerical representation of incomparables (infinitesimals and infinitely great but still 'hyperfinite' reals) added by hyperreal models have never had even a well-established and respectable *geometrical* presence. And there never seemed to be any danger that, in working with the field of reals as opposed to the field of rationals, mathematicians had to worry about 'losing' any structures thought to be mathematically significant. Therefore, there was no compelling *mathematical* reason for the nineteenth-century real analyst (or 'rational analyst') to adopt a Janus-faced perspective, keeping in view both ordered fields ℚ and ℝ *as separate but related models*. While eminently useful in terms of the current mathematical employment of non-Archimedean models, the Janus-faced perspective introduces, as I shall argue in the next section, a sort of ambivalence into the attempt to use non-Archimedean models *conceptually*–e.g., the attempt to use such models in order to resolve certain classical problems concerning continua. I conclude this talk with several examples of why the attempt to use non-Archimedean models in this conceptual or philosophical way proves problematic.

V. Two Examples of the Attempt to Use Nonstandard (Non-Archimedean) Models Philosophically.

A. Zeno's Dichotomy Paradox: One Resolution.

According to Zeno's Dichotomy paradox, a runner is charged with traversing a distance of unit length. But, according to Zeno, before the runner can reach the goal point–call it '1'–he must first traverse, in order, an infinite sequence of checkpoints $\{(2^n - 1)/2^n\}$, beginning with $n = 1$ and continuing as n increases without limit, i.e., {1/2, 3/4, 7/8, 15/16, 31/32, . . . }. According to one (charitable) interpretation, Zeno's principal point is that such a runner cannot reach the goal 1 because he *first* has to complete, sequentially, an infinite number of actions (each one associated with reaching a checkpoint and no one of which is equivalent to reaching the goal 1) of which there is no last member–and that, Zeno believes, is impossible. According to another version of the Dichotomy, the sequence of tasks of reaching checkpoints is inverted, {. . . 1/32, 1/16, 1/8, 1/4, 1/2, 1}, so that there is no first member of the sequence and the runner, according to Zeno, cannot 'get started'.

H. Jerome Keisler outlines a resolution of the Dichotomy that makes use of a non-Archimedean concept to which I have previously alluded, a *hyperfinite grid.* Where H is an infinite hypernatural number, the hyperfinite grid $\mathbb{H}$ with mesh $1/H$ is the set of all multiples of $1/H$ between $-H$ and H. As Keisler notes, an H is usually selected such that every standard natural number divides it (e.g., by letting $H = J!$ or J factorial, for some infinite hyperfinite number J), with the result "that each standard rational number belongs to $\mathbb{H}$, that is $\mathbb{Q} \subseteq \mathbb{H}$." Keisler applies such a hyperfinite grid, restricted to the unit interval [0, 1], to Zeno's Dichotomy:

> On the hyperfinite grid, Zeno's [Dichotomy] Paradox is resolved as follows. We can get from 0 to 1 in H steps by taking one step of length $1/H$ every $1/H$ seconds, always staying in the hyperfinite grid $\mathbb{H}$. Along the way, we will pass through all the points 1/2, 3/4, 7/8, and so on, since they all belong to the set $\mathbb{H}$. Of course, we will overshoot irrational points such as $\sqrt{2}/2$, but there will be a time at which we pass from below $\sqrt{2}/2$ to above $\sqrt{2}/2$ with one step of length $1/H$.

Such a hyperfinite grid, as well as its restriction to the unit interval, are 'hyperfinite'–that is, they belong to the extension of the predicate 'is finite' when interpreted in the non-

Archimedean superstructure model V(*R) of hyperreal numbers. As Keisler notes, such a hyperfinite subset of *R "inherits the first order properties of finite subsets of R," as well as appropriately weakened higher-order properties. Thus, for example, there is no *internal* one-to-one mapping between such a set and any of its proper subsets. Also, in terms of the < relation, there is a 'first' member and a 'last' member of each such set, and its members are discretely ordered: that is, any member that has a successor (predecessor) has a unique, immediate successor (predecessor). Consequently, many of the worries about the possibility of completing, sequentially, an infinite sequence of tasks seem to disappear. Although from an 'external' (standard) point of view, the runner must complete an infinite sequence of tasks, there will be a first task and a last task for him to complete; and any task strictly between these will have a unique, proper succeeding and a unique, proper preceding task, etc. And, as Keisler notes, each of the Zenonian checkpoints in the sequence $\{(2^n - 1)/2^n\}$ is 'embedded' as one of these tasks–as is each of the checkpoints in the inverted Zenonian sequence {. . . 1/32, 1/16, 1/8, 1/4, 1/2, 1}.

Does, then, Keisler's resolution 'work'? *Sic et non.* There are subsets of the hyperfinite grid restricted to [0, 1] that raise some of the same Zenonian worries about completing sequentially an infinite sequence of tasks. For example, there is the set of all initial steps, individually of length $1/H$, the sum of which is *less than* any real value, as well as the complement of this set relative to the hyperfinite grid restricted to [0, 1]. It would seem that all of the tasks or steps in the former set would have to be completed, sequentially, before he could begin work on the tasks/steps in its complement. But there is no last task/step in the former set nor any first task in the latter. These 'problem-causing' subsets, however, must be *external* subsets of the hyperfinite grid restricted to [0, 1]. This means that they do not fall within the range of the higher -order bounded quantifiers over sets of hyperreals (that is, quantifiers ranging over *℘(R)), when the wffs of the theory of the reals are interpreted in the non-Archimedean superstructure V(*R). Does it also mean that the sets are 'not there' to cause problems for Keisler's analysis of the Dichotomy? Well, in most formulations of non-Archimedean analysis such problem-causing sets *are* 'there', as external sets or members of ℘(*R) - *℘(R). Although this may be a sort of mathematically 'ghostly' existence (in terms of the interpretation of the theory of reals in the nonstandard model V(*R)), it is far from clear (to me, at least) that the existence of such subsets does not raise more-or-less the same issues about the possibility of completing, sequentially, an infinite sequence of tasks, that Keisler's resolution of the Dichotomy in terms of hyperfinite grids was supposed to avoid.

B. Zeno's Dichotomy Paradox: Another Resolution.

Perhaps the most interesting attempt with which I am familiar to apply non-Archimedean concepts to a classical philosophical problem is the 'critical' application of Nelson's IST to Zeno's Dichotomy paradox by William I. McLaughlin and Sylvia L. Miller. The argument of McLaughlin and Miller depends on an epistemological assumption

> (E2) The fact that an object is located at a point in spacetime cannot be established if the coordinates describing the point are nonstandard real numbers.

McLaughlin and Miller add that "the phrase '[t]he fact that' in E2 means that the object's location has been observationally verified or could have been observationally verified had one been sufficiently equipped and attentive to capture the requisite numerical description of the event." In a later manuscript by McLaughlin, this assumption becomes the "critical mensuration thesis," which he characterizes as follows:

> every phenomenon can be completely described through the use of real numbers, but not all real numbers can be used for describing phenomena. The first clause, the "mensuration thesis", in the statement of the greater thesis, rests upon the success of experimental science. The second clause must be argued, and this is carried out through the medium of internal set theory. . . .

Since McLaughlin is working within the framework of Edward Nelson's nonstandard set theory–Internal Set Theory (IST)–which does not provide for the existence of external sets, the extension of the predicate 'real number', as he uses the phrase, is what is designated *R in other formulations, i.e., the hyperreal numbers, and includes nonstandard infinitesimals, infinitely large reals, etc. As it turns out, the second clause will rule out precisely the nonstandard reals for use in describing physical processes:

> Although the mensuration thesis has appealed to real numbers for the means to express results of a measurement process, it is clear that the thesis can be extended to other mathematical objects which serve as measurement labels, e.g., complex numbers,

> vectors, real intervals. Specimens of such objects would be suitable for the process only if they were standard.

And, says McLaughlin, "we consistently adopt the perspective of an observer who is measuring phenomena and have shown that nonstandard numbers are not available as measurement labels for those phenomena."

Now, consider the set of 'checkpoints' of Zeno's Dichotomy, designated by McLaughlin and Miller as $C = \{r \in [0, 1]: r = 1 - 2^{-j}, 1 \leq j < \infty\}$. From the perspective of IST, this (standard) set will contain (a great many) elements that are nonstandard. That is, it will include infinitesimals for (all) values of $r = 1 - 2^{-j}$ where j is what is variously called an 'infinite hypernatural' or 'illimited [positive] integer' (which numbers possess, of course, the first-order properties of standard *finite* natural numbers). But, then, by the critical mensuration thesis (hereafter, CMT), the set will contain (nonstandard) elements that cannot describe any physical process and, consequently, no runner could correctly be described as traversing the set C of spatial loci.

But what of the set frequently characterized, in some formulations of nonstandard set theory, as ^{o}C or ^{0}C, the set of all and only the *standard* elements of the (standard) set C? We might–perhaps with some prejudice–characterize this as the 'original set giving rise to the Zenonian worry'. In other formulations of non-Archimedean mathematics it would be an external set. But in Nelson's IST, although all of its members can be said to exist *individually* as standard mathematical entities, the set ^{o}C simply does not exist. In fact, IST does not allow for the existence of *any* set that is infinite (in the usual, standard sense of 'infinite') but contains only standard elements.

So, the supposed infinite sequence of tasks of Zeno's Dichotomy cannot be described by the (nonexistent) set ^{o}C; and application of the CMT rules out its description by the set C, which contains nonstandard members. Consequently, McLaughlin draws a finitistic conclusion about (the mathematical description of) a physical process such as Zeno's Dichotomy:

> For Zeno's Dichotomy, incursion into an infinitesimal neighborhood of 1 was seen to be possible but epistemologically opaque, disabling his claim of paradox. . . . Assume that there are no physical constraints to prevent a traverse of any finite segment of the Checkpoint sequence of The Dichotomy, and . . . implement a counting scheme to register

the passage of the moving object past each checkpoint. It must be the case that the count ceases prior to the recording of an unlimited natural number. This implies that the count must terminate at some standard natural number. . . ; to avoid paradox, the premise of no-physical-constraints must be judged false. That is, phrased positively, *physical* reasons must prevent the observations from being made.

McLaughlin's argument, perhaps somewhat oversimplified, is the following: Physical-epistemological considerations preclude any mathematical description of a physical process that appeals to a set containing nonstandard numbers. But IST provides only for *finite* sets of elements containing nothing but standard elements. Therefore, the mathematical description of physical processes must be finitistic–apparently, suggests McLaughlin, because of physical-epistemological reasons.

McLaughlin does not intend the CMT to yield such a finitistic conclusion *directly*. The use of IST–and in particular, its 'disappearance' of external sets–is a crucial step in his argument. Consequently, it seems to me that one cannot infer that the finitisitic conclusion of his argument is due *exclusively* to physical-epistemological considerations. And, I think, some interpretative qualification is required for the following comment by McLaughlin: "the premise of non-physical-constraints must be declared false" and "*physical* reasons must prevent the observations [of checkpoints beyond some finite number] from being made." The *combination* of the CMT and IST yields such conclusions. If one were to accept IST + CMT, that fact would entail that any physical theory could not fail to be finitistic, "on pain of being mathematically unintelligible," as McLaughlin has expressed it to me. However, I suspect that the appeal of any physical theory having a finitistic character–and, indeed, the persuasiveness of arguments on behalf of such a theory–would rest on considerations quite independent of IST + CMT.

The fact that Nelson's IST is, in its elimination of external sets, less Janus-faced than competing formulations–including other set-theoretic formulations–of non-Archimedean mathematics seems to be crucial to McLaughlin's argument. From the perspective of alternative such formulations providing for the existence (as an external set) of the 'original' Zenonian set $^{\sigma}C$ of checkpoints, there would be no apparent reason to disallow this set as an acceptable description of a sequence of actions performed by Zeno's runner, since McLaughlin's CMT *in* *itself* does not rule it out. But then we are faced afresh with all the puzzles concerning the sequential performance of an infinite sequence of acts. One could, of course, invoke physical-epistemological considerations in order to strengthen the CMT so that it *does* commit one to finitistic descriptions of physical processes. But that would be to render the IST otiose in the argument for such a conclusion.

VI. A Very Brief Conclusion.

Although the development of non-Archimedean mathematics has provided incomparables with a spacious numerical *Lebensraum* in the hyperreal line, there is, I think, little likelihood of a 'paradigm-shift' from the real to the hyperreal line. The fact that most of contemporary non-Archimedean mathematics has been developed as a conservative extension of the relevant classical Archimedean mathematical theories has given it the Janus-faced character that militates against such a paradigm shift. Moreover, I have suggested that this Janus-faced character has made non-Archimedean mathematics of doubtful use, at present, in addressing 'deep' conceptual or philosophical issues pertaining to continuity and infinity.

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