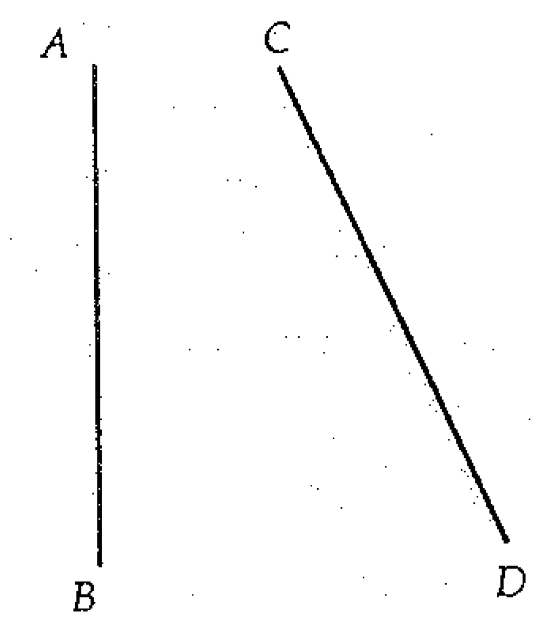


ergo dicendum: Relationum, quae sunt Entia, tum vera, cum a nobis cogitantur, ut sunt numeri, lineae seu distantiae, aliaque id genus; non esse numerum, nam perpetuis semper reflexionibus possunt multiplicari, adeoque nec sunt Entia realia, possibiliave nisi cum cogitantur. Quaeritur an ullus sit numerus, qui desit in mundo, videtur id non posse fieri, ubi enim obsecro subsisteremus, dicendum scil. id intelligendum ita tamen, ut eaedem res in diversos aliquando numeros intrent. Nam alioqui si semper id fieri posset ope novarum rerum, tunc tot essent res [quot] numeri, quod impossibile, quia aliqua certa est multitudo rerum, nulla est Numerorum.

Modificationes existentes sunt illae, quae possibiles; idem [—BREAKS OFF.]

Numeri infiniti Aiii69

496 Si duo Numeri infiniti, sint homogenei, id est ut unus certo numero finito multiplicatus alterum superet; sive qui sint inter se, ut duae lineae designabiles; tunc si sint commensurabiles, erunt, ut numerus finitus[L1] ad numerum finitum.[L2] Sint duae lineae infinitae *AB etc. CD etc.* Utraque dividi posse intelligatur exacte in pedes *AI, IF etc.* vel *CG, GH etc.* sive intelligatur utramque ex multitudine pedum conflari ita, ut nulla pedis fractio supersit, tunc manifestum est has duas lineas, licet infinitas, esse commensurabiles, sive habere communem mensuram, pedem scilicet *AI.*

L1. ABOVE numerus finitus LEIBNIZ WROTE: Error.
L2. ABOVE numerus finitus: error.

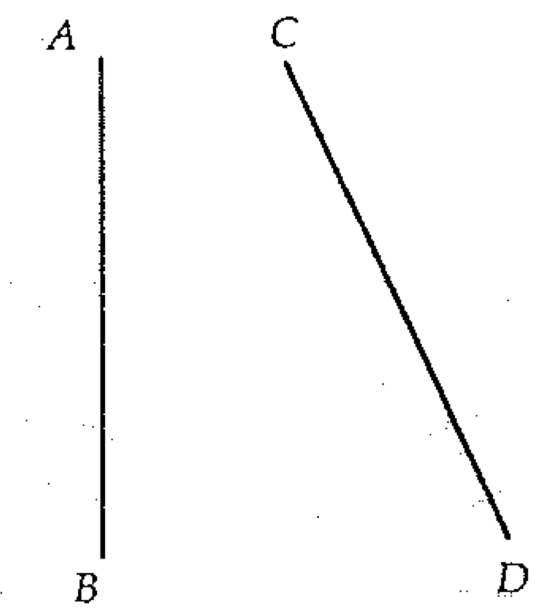


Figure 3

Therefore it seems that what should be said is this: there is no number relations, which are true entities only when they are thought about by ι for example, numbers, lines, or distances, and other things of that kind; f they can always be multiplied by perpetually reflecting on them, and they are not real entities, or possibles, except when they are thought abo Supposing it is asked whether there is any number which ceases to exist the world, it seems that this is impossible: for as long as (pray) we subsi it must be said that the number would have to be understood, but in sucl way that the same things would sometimes enter into different numbe For otherwise, if it could happen that there were always a need for ne things, then there would be as many things as numbers; but this is impo sible, because the multiplicity of things is something determinate, that numbers is not.[18]

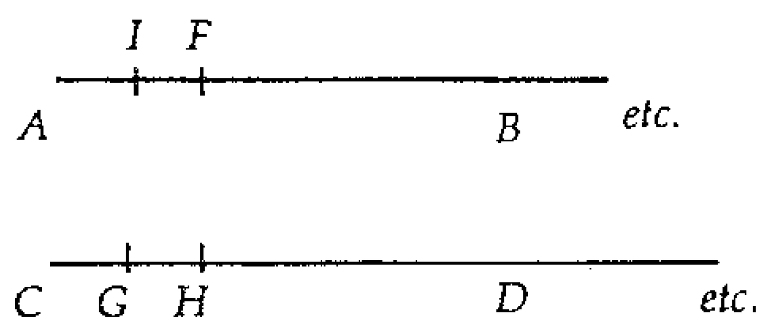
Modifications that exist are those which are possible; likewise (—BREA OFF).

16. Infinite Numbers[1] Aiii

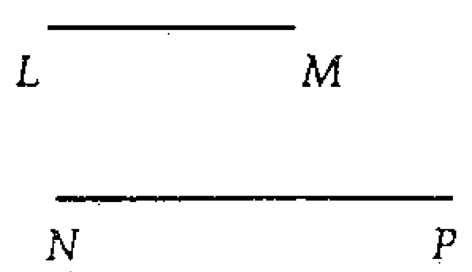
[c. 10 April 1676][2]

496 If two infinite numbers are homogeneous, i.e. such that one multiplied a certain finite number exceeds the other; or, are to each other as two de ignatable lines; then if they are commensurable, they will be as finite number to finite[L2] number. Let there be two infinite lines *AB* . . . a *CD*. . . . Let both be understood to be divisible exactly into feet, *AI, IF, e* and *CG, GH, etc.*; that is, let both be understood to be made up of a mul plicity of feet in such a way that no fraction of a foot is left over. Then i evident that these two lines, though infinite, are commensurable, i.e. ha

L1. ABOVE THIS: Error.
L2. AGAIN, ABOVE THIS: error.

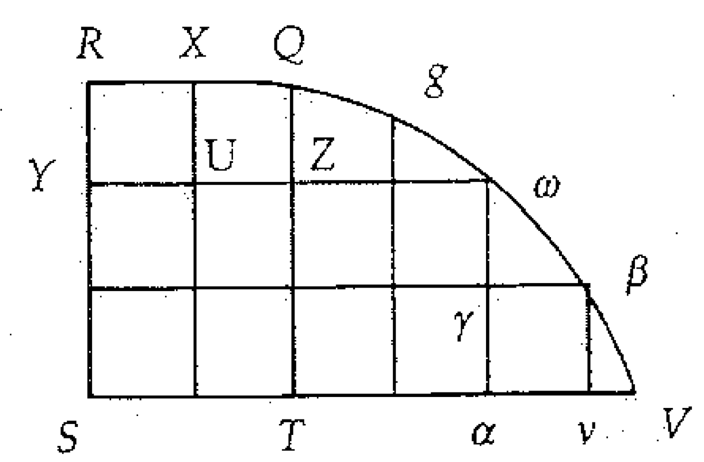


Intelligatur jam has ipsas duas lineas, licet infinitas, esse finitae inter se invicem rationis, seu ut lineae finitae *LM* et *NP* sive homogeneas, id est tales, ut aliquoties repetita minor (aliquoties, finito scilicet repetitionum numero) majorem excedat; tunc manifestum est has duas lineas fore inter



se, ut numerus finitus ad numerum finitum. Sunt enim ut duae lineae *LM* et *NP*, et ipsae sunt commensurabiles. Ergo hae duae lineae *LM* et *NP* sunt commensurabiles; duae autem lineae finitae commensurabiles sunt ut numerus finitus[L3] ad numerum finitum.[L4] Ergo et duae lineae propositae *AB etc.* et *CD etc.*

Sint jam duae figurae, una rectilinea *QRST*, altera curvilinea mixta *QTVQ*, eiusdem altitudinis *QT*. Ponatur curvilinea esse polygonum *QTvβγω*



seu figura gradiformis, divisa in infinita Quadrata infinite exigua qualia
497 *αvβ[γ]* eodem modo figura rectilinea in quadrata aequalia prioribus qualia

L3. Above numerus finitus: error.
L4. Above numerus finitus: error.

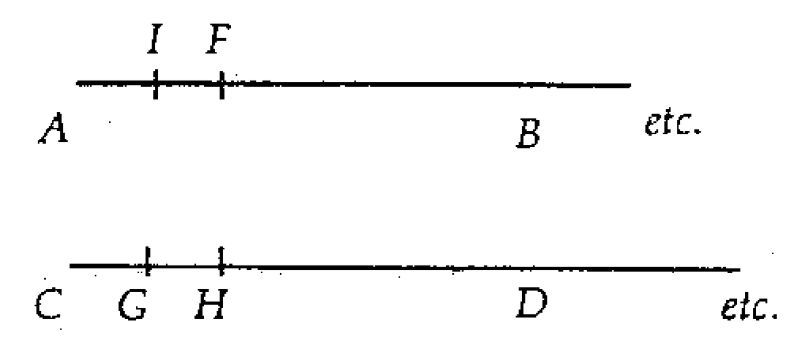


Figure 1

a common measure, namely, the foot *AI*. Now let it be understood tha these same two lines, though infinite, have a finite ratio to each other, i.e are as the finite lines *LM* and *NP*. In other words, let them be *homogeneous,* i.e. such that the smaller, when repeated several times ('severa times' meaning 'by a finite number of repetitions'), exceeds the greater

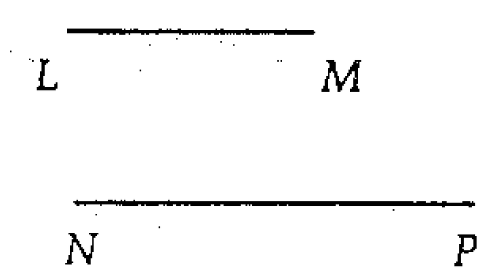


Figure 2

then it is evident that these two lines will be to each other as finite numbe to finite number. For they are as the two lines *LM* and *NP*, and they ar commensurable. Therefore these two lines *LM* and *NP* are commensurable; but two commensurable finite lines are as finite[L3] number to finite[L] number. Therefore so are the two lines proposed, *AB* . . . and *CD* . . . .

Now let there be two figures, one rectilinear *QRST*, the other a mixe curvilinear one *QTVQ*, of the same height *QT*. Let us suppose that th curvilinear one is a polygon *QT*vβγω, that is, a gradiform figure divide into an infinity of infinitely small squares, such as αvβ[γ].[3] In the sam

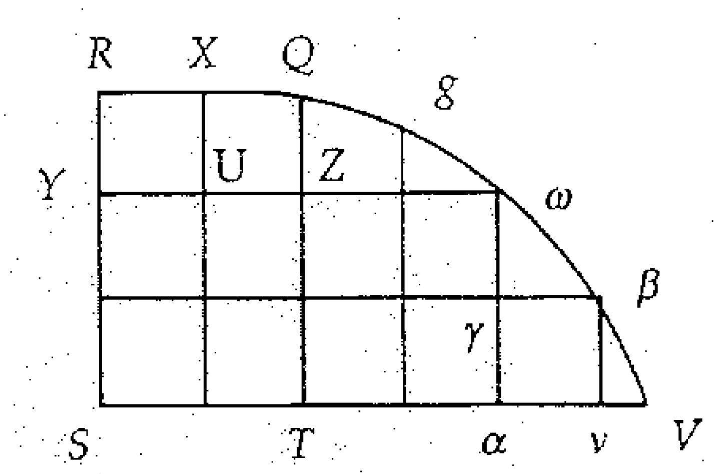


Figure 3

way, let the rectilinear figure be understood to be divided into square equal to these, such as *XQZ[U]*. In this way, the height *QT* is understood t
497 be divided into an infinity of parts, and however many ordinates the recti

L3. Again, above this: error.
L4. Again, above this: error.

XQZ[U] divisa intelligatur; hoc modo et altitudo *QT* divisa intelligetur in partes infinitas; et quaelibet ordinata tam figurae rectilineae, ut *ZY*, quam curvilineae, ut *Zω*. Ponatur autem quaelibet ordinata curvilineae, ut *Zω*, esse rationalis ad ordinatam respondentem rectilineae, ut *ZY*, quod fieri poterit, si ita ferat aequatio curvae; erit et numerus quadratorum infinitorum unius commensurabilis, numero quadratorum alterius, si una exhauritur, exhaurietur et altera repetitione quadratorum; cumque singulae ordinatae singulis hoc modo habeant communem figuram, etiam totae figurae habent communem mensuram, quadratum scilicet assumtum; si ordinatae hae omnes in rectum extendantur, seu quadrata in directum ponantur, erit eorum numerus ex figura rectilinear ad numerum e curvilinea, ut linea quaedam infinita, commensurabilis, ad infinitam commensurabilem, adeoque ut supra ostendimus, ut numerus finitus ad numerum finitum. At ejusmodi curvilinea est circulari σύλλογος, ut alibi ostendi, ergo circulus ad quadratum, ut numerus finitus ad numerum finitum, quod est absurdum. Et jam video rationem erroris. Negandum duas lineas commensurabiles finitas esse ut numerus finitus ad numerum finitum. Possunt esse, ut numerus infinitus ad infinitum. Duo numeri infiniti commensurabiles possunt esse qui non sint ut duo numeri finiti; si scilicet maxima eorum communis mensura sit numerus finitus. Ut si ambo sint primi. Illud interea hic certum est, has duas figuras circulum et quadratum esse commensurabiles, seu habere mensuram communem, sive finitam et ordinariam (quo casu essent, ut numerus finitus ad numerum finitum, quod minime conciliabile arbitror cum appropinquationibus), sive infinite parvam; quod necessarium arbitror.

Hinc jam tandem videtur aditus apertus ad mirabilem demonstrationem, quod quadratura circuli sit impossibilis, qualis quaeritur; quae scilicet aequatione aequabili exprimat relationem. Quod ut fiat ostendendum est, quod Diameter et latus, ne infinite quidem parvam habeant communem mensuram, et in genere linea quae sit ut radix irrationalis, sive
498 quadratica, sive cubica; ut exempli causa, latus cubi dupli; alteriusve. Ecce hinc praeclarum usum demonstrationum, de incommensurabilibus linearium, possunt enim et ad infinite parva transferri, quod non possunt arithmeticae. Hoc posito, sequitur circuli magnitudinem non posse aequatione quadam ullius gradus exprimi. Eodem argumento evincitur ne ullam quidem circuli portionem hoc modo quadrari posse; idemque est de Logarithmis et Hyperbola.[L5] Circulus aliaque id genus, Entia ficta sunt; ut

L5 In the margin: Huic ratiocinationi, quae probare videtur, Circulum non esse quadrabilem, tamdiu non est fidendum, quamdiu non est evictum, Diagonalem non

linear figure has, e.g. *ZY*, so does the curvilinear one, e.g. *Zω*.[4] Now let u assume any arbitrary curvilinear ordinate, such as *Zω*, to be in rational pro portion to the corresponding rectilinear ordinate, *ZY*, which will be possi ble if the equation of the curve allows it; then the number of infinite[simal squares of one will also be commensurable with the number of squares o the other, and if one is exhausted by a repetition of squares, then so will th other be. And since in this way the individual ordinates taken one by on have a common figure, so the whole figures will also have a common mea sure, namely, the assumed square. If all these ordinates are extended in straight line, i.e. the squares are taken directly, the number of them in th rectilinear figure will be to the number in the curvilinear one as one com mensurable infinite line to another, and so as we showed above, as finit number to finite number. But a curvilinear figure of the above kind is con gruent with the circle, as I have shown elsewhere. Therefore the circle is t the square as finite number to finite number, which is absurd.

And now I see the reason for the error. It must be denied that two com mensurable finite lines are as finite number to finite number. It is possibl for them to be as infinite number to infinite.[5] Two infinite numbers whic are not as two finite numbers can be commensurable, namely, if thei greatest common measure is a finite number—for instance, if both ar prime.[6] Meanwhile, this much is certain here, that these two figures, th circle and the square, are commensurable, i.e. have a common measure whether (i) finite and ordinary (in which case they would be as finite num ber to finite number, which I believe to be completely irreconcilable wit approximations); or (ii) infinitely small, which I believe to be necessary.

Hence now at last there seems to be a way open for a marvelous demon stration that it is impossible for there to be a quadrature of the circle of th kind we are seeking: namely, one which would express the relation by a equable equation. And in order for this to be done, it must be shown tha the diameter and the side do not have even an infinitely small commo measure, even in the kind of line which is as an irrational root, whethe
498 quadratic or cubic—e.g. the side of a double cube [$\sqrt[3]{2}$]—or of som higher power. Hence here we have a splendid use for demonstration about incommensurables using lines, for they can also be carried over t the infinitely small, which those of arithmetic cannot. Supposing this, i follows that the magnitude of a circle cannot be expressed by an equatio of any degree.[8] By the same argument it is proved that not even a portio of a circle can be squared by this means; and it is the same with the loga rithm and the hyperbola.[L5]

L5. In the margin: This reasoning, which seems to prove that a circle is no squarable, should not be relied on as long as it has not been proved that the diago

Polygonum, quolibet assignabili maius, quasi hoc esset possibile. Itaque cum aliquid de Circulo dicitur, intelligimus id verum esse de quolibet polygono, ita, ut aliquod sit polygonum, in quo error minor sit quovis assignato *a,* et aliud polygonum in quo error minor alio quolibet certo assignato *b.* Non vero erit polygonum, in quo is sit minor omnibus simul assignabilibus, *a* et *b,* etsi dici possit, ad tale ens quodammodo accedere polygona ordine; itaque si polygona certa quadam lege crescere possint; et de iis aliquid verum sit quo magis crescunt, mens nostra quiddam ultimum fingit; deque eo id quod in singulis magis magisque evenire videt, perfecte pronuntiat, quod etsi non sit in rerum natura, ferri tamen eius expressio potest; compendiosarum enuntiationum causa.

Caeterum videndum est an non sint adhuc alia, infinite parva, ut anguli. Ecce enim angulus, nonne in puncto est. Nihil enim ad eum pertinet laterum longitudo, et manet etsi semper abscindas. Ergo quantitas in puncto, nam anguli quantitas est. Respondendum primum angulum in solo puncto nullum esse, nisi accedant lineae. Si jam eae lineae sint infinite parvae, lineae tamen, manebit difficultas, eodem enim modo ab illis resecabo. Porro non est angulus quantitas puncti. Posuimus enim punctum cuius pars nulla est, extremum scilicet; aliud enim inassignabile, nullum esse, jam ostendimus. Erit ergo anguli quantitas nihil aliud quam quantitas sinus proportionalis, quae utcunque producas eadem, ita ut ipse angulus videatur esse Ens fictitium. Si pro aliqua re in ipso puncto existente sumatur, sci-
499 licet si ponamus, Angulus est in puncto, seu linea quavis assignabili resecta subsistit, datur quantitas anguli. Idem est angulus, lateribus productis. His positis erit angulus, id quod est in lineis qualibet intersecta minoribus, seu spatium comprehensum duabus lineis concurrentibus, qualibet assignabili minoribus. At tale Ens fictitium est, quoniam lineae eiusmodi fictitiae.

Etsi Entia ista sint fictitia, Geometria tamen reales exhibet veritates, quae aliter, et sine ipsis enuntiari possunt, sed Entia ilia fictitia praeclara

posse, detracta saltem quantitate infinite parva, tunc reddi commensurabilem lateri, sumta mensura infinite parva. Idemque esse de caeteris radicibus.

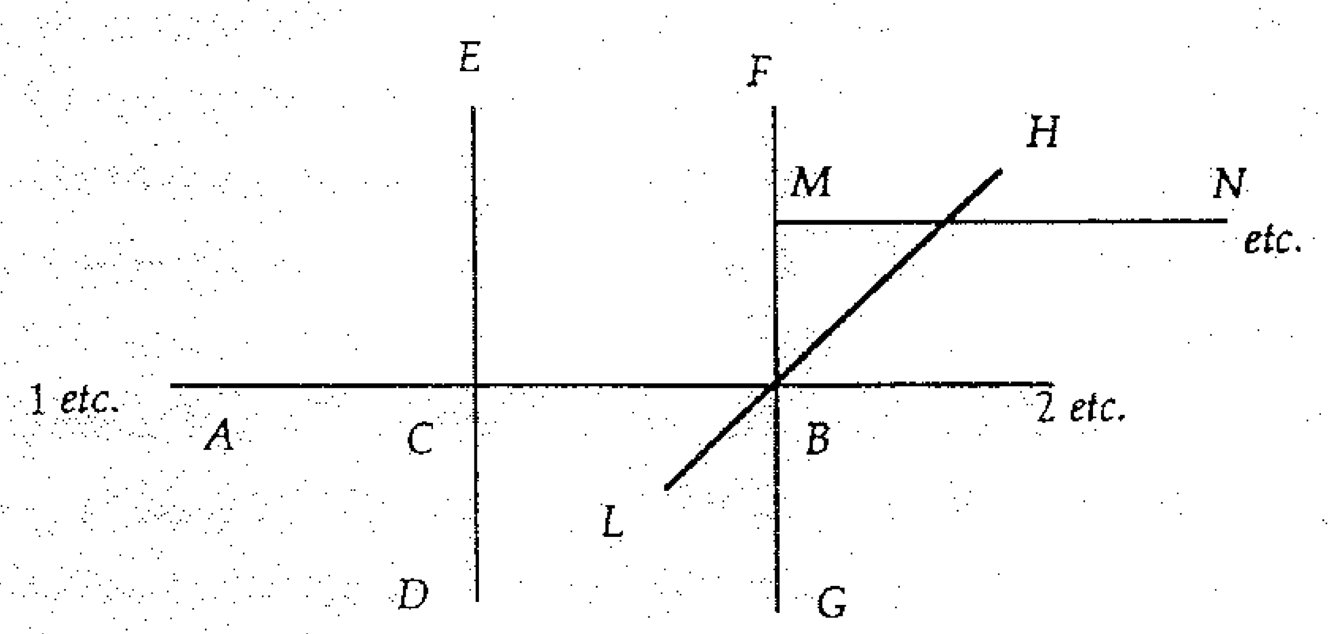
The circle—as a polygon greater than any assignable, as if that were possible—is a fictive entity, and so are other things of that kind. So when something is said about the circle we understand it to be true of any polygon such that there is some polygon in which the error is less than any assigned amount *a,* and another polygon in which the error is less than any other definite assigned amount *b.*[9] However, there will not be a polygon in which this error is less than all assignable amounts *a* and *b* at once, even if it can be said that polygons somehow approach such an entity in order. And so if certain polygons are able to increase according to some law, and something is true of them the more they increase, our mind imagines some ultimate polygon; and whatever it sees becoming more and more so in the individual polygons, it declares to be perfectly so in this ultimate one.[10] And even though this ultimate polygon does not exist in the nature of things, one can still give an expression for it, for the sake of abbreviation of expressions.

For the rest, it must be seen whether there are not still other things that are infinitely small, such as angles. Here, for instance, is an angle; is it not in a point? For the length of the sides is irrelevant to it, and it remains even if they are shortened forever. Therefore there is a quantity in a point, for it is the quantity of an angle.[11] First it should be replied that there is no angle in a point by itself unless lines are added. Now if these lines are infinitely small, but are lines, the difficulty will remain, for I will cut them back in the same way. Moreover, an angle is not the quantity of a point. For we have supposed a point to be that whose part is nothing, an extremum; for we have already shown that there is nothing else unassignable. Therefore the quantity of the angle will be nothing but the quantity of the proportion of the sine, which is the same however far you might produce it, so that the angle itself, it seems, is a fictitious entity. If we take it for some thing ex-
499 isting in the point itself, that is, if we suppose that the angle is in the point, i.e. that it subsists when any assignable line has been cut back, the quantity of the angle is given. It is the same as the angle with the sides produced. If these things are assumed, an angle will be that which is in lines smaller than any that intersect, i.e. the space comprised by two intersecting lines that are smaller than any assignable. But such an entity is fictitious, since lines of this kind are fictitious.

Even though these entities are fictitious, geometry nevertheless exhibits real truths which can also be expressed in other ways without them. But

nal cannot—at least, by subtracting an infinitely small quantity—be rendered commensurable to the side, assuming an infinitely small measure. And the same holds for the other roots.

sunt enuntiationum compendia, vel ideo admodum utilia, quia imaginatio nobis Entia eiusmodi apparere facit id est polygona quorum latera non distincte apparent, unde nobis suspicio oritur postea Entis nulla latera habentis. Quid vero an non imago illa saltem nulla exprimit polygona? Ergo imago illa menti perfectum exprimit Circulum. Est hic quaedam difficultas mira et subtilis. Etsi enim falsa sit imago, in se tamen Ens est verum; adeoque sequitur in mente esse circulum perfectum, vel potius esse imaginem realem. Erunt ergo in mente et caetera omnia: et in ea omnia jam fient, fieri quae posse negabam. Sed dicendum est in mente esse cogitationem uniformitatis, nullam autem circuli perfecti imaginem, sed a nobis applicari uniformitatem postea ad hanc imaginem, quod nos sensisse inaequalitates, obliviscimur; consciine aliquando fuimus nos sensisse? hoc enim necesse ad oblivionem. Sed hoc non est. Dicendum ergo potius, cum circulum sentimus, vel polygonum, nunquam nos in eo sentire uniformitatem, sed saltem non sentire difformitatem, seu meminisse nos nihil in eo difforme sensisse; quoniam inaequalitas non statim percellebat oculos. Et ob hanc memoriam ipsi jam uniformitatis nomen tribuimus. Videndum an non multorum per exigua temporis intervalla conscii simus, quorum non meminimus, seu de quibus non possumus loqui, scribere, quae non possumus characteribus exprimere, ob eorum exiguitatem, cum parvam habeant ad talia relationem. Non ideo tamen minus sentiuntur a nobis consciis. Sed eorum, quemadmodum eorum quae somniamus, obliviscimur.



Examinandum adhuc restat nonnihil de Lineis interminatis. Primum in linea interminata necesse est esse punctum medium. Sit in ea punctum C
500 alicubi, jam sunt aut aequales aut inaequales lineae *Cl etc.* et *C2 etc.* Si ae-

these fictitious entities are excellent abbreviations for expressions, and for this reason extremely useful. For entities of this kind, i.e. polygons whose sides do not appear distinctly, are made apparent to us by the imagination, whence there arises in us afterwards the suspicion of an entity having no sides. But what if that image does not represent any polygons at all? Then the image presented to the mind is a perfect circle. Here there is a surprising and subtle difficulty. For even if the image is false, the entity in it is nevertheless true; and so it follows that in the mind there is a perfect circle, or rather, there is a real image. Therefore everything else will also exist in the mind: and in it everything that I denied to be possible will now be possible. Instead, what must be said is that in the mind there is a thought of uniformity, yet no image of a perfect circle: instead we apply uniformity to this image afterwards, a uniformity we forget we have applied after sensing the irregularities. Were we then conscious at some time that we had sensed them? for consciousness is necessary for forgetting. But this is not the case. Therefore it must be said, rather, that when we sense a circle or polygon, we never sense uniformity in it, but neither do we even sense a nonuniformity, that is to say, we do not remember having sensed anything nonuniform in it, since the inequality did not immediately strike us. And because of this memory we now ascribe the name of uniformity to it. It must be seen whether we might not be conscious for very small intervals of time of many things we do not remember, or about which we are unable to speak or write, which we cannot express in characters on account of their extremely small size, since they would have little relation to such things. But they are not on this account any less sensed by our consciousness. Rather, we forget about these things, just as we forget about the things we dream about.[12]

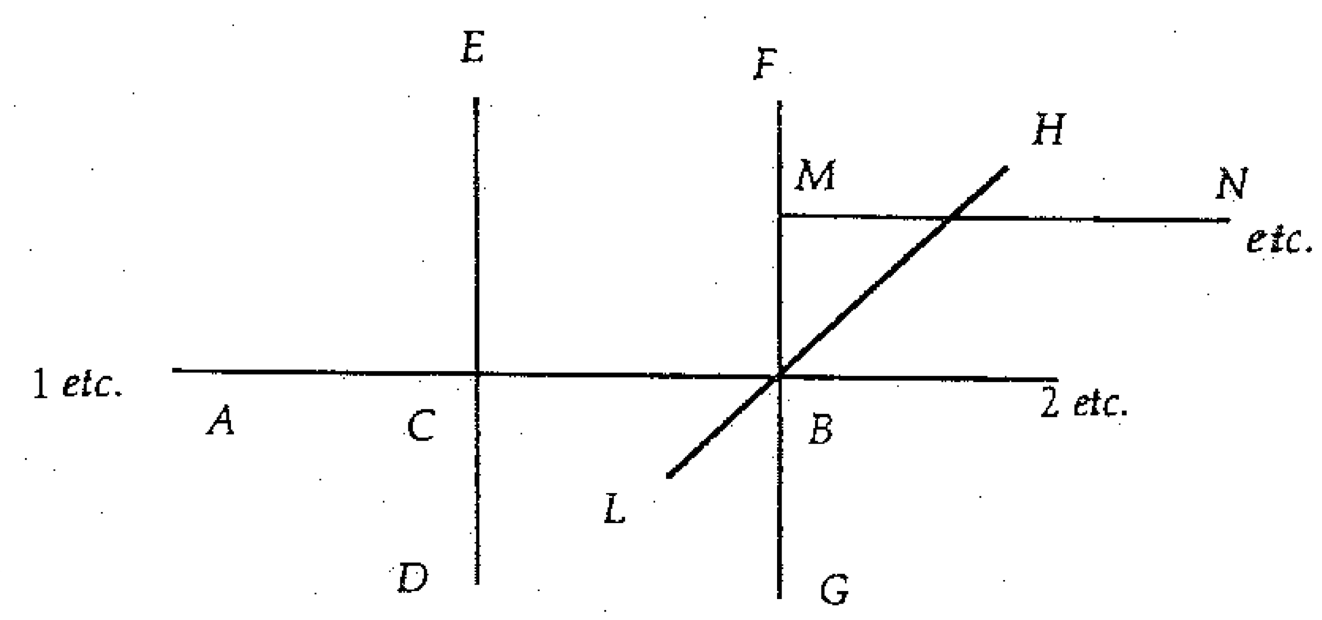


Figure 4

Something still remains to be examined concerning unbounded lines.[13] First, in an unbounded line there is necessarily a midpoint. Let there be a
500 point C somewhere in it. Now the lines *C1 . . .* and *C2 . . .* will either be

quales, punctum *C* erit medium. Si inaequales, alterutra earum erit major. Equidem posse videor statim assumere, ita licere lineam *1 etc. 2 etc.* interminatam utrinque secare, ut partes duae sint aequales, in *B* scilicet, et tunc *B* erit medium, sed videndum an hoc amplius probari possit. Sit ergo major *[2] etc.*, minor *1 etc.*: ajo primum non posse esse unam alterius triplam. Quoniam in tres interminatas ab uno latere non resolvi potest interminata utrinque, sed in duas tantum; eademque ratione ostendi potest, nec rationem posse exprimi fractione numeris finitis assignabili media, inter *2* et *3* eandem ob causam. Itaque fractionis illius numerus infinite parva quantitate excedet *2*. Sin linea interminata una *C2 etc.* alteram *CA1 etc.* finita excedet quantitate, qualis e. g. *CB*, ipsa ergo assignata et definita, erit *B* punctum medium. Hinc jam lineae *ED* per *C* transeuntes, non erunt in medio universi, sed *FBG* erit. Ponatur eodem modo et *B* punctum esse medium ipsius *FG* interminatae, et ipsius *LH* interminatae, aliarumque omnium inde ductarum, erit ergo punctum *B* medium universi. Videndum autem an necesse sit inveniri eiusmodi medium posse. Sed hinc videtur sequi difficultas; ducatur recta *MN etc.* interminata versus *N;* erit illa aequalis ipsi *B2 etc.*, cum sit parallela, certum est enim illam infinitam sibi parallelam ferri posse angulo ad se ipsam recto seu in recta *BF.* Videndum jam an ipsi *BH* sit aequalis an inaequalis; major an minor. Si *BH* aequalis, seu si aliquod Medium universi, uti aliunde videtur judicari posse, dicendum erit *BH* aequari *MN.* Hinc jam patet, quomodo moveri possit linea infinita, seu transferri *BH* in *MN,* non continuo motu, id enim impossibile est, sed per saltus, si scilicet *transcreetur.* Est enim motus nihil aliud, quam transcreatio.^{L6}

Sed una restat magna *difficultas*. Sit linea infinita, vel utrinque, vel ut lubet, ab una parte, quae actu resoluta ponatur in suas partes; *etc. AB, BC, CD, DE, etc.* Potest enim intelligi innumera eiusmodi esse Entia. Unum-
501 quodque eorum moveri potest, unum ex *AB* in [αβ,] alterum ex *BC* in βγ, tertium *CD* in γδ, quartum ex *DE* in δε, quot cum in singulis sit possibile, singula translata intelligamus; dici poterit totam lineam paralleliter sibi et

L6. In the margin: Forte hinc probari potest, omnem conatum esse in rectis perpendicularibus aut coincentibus. Idque experientia confirmat, ut in Tangentibus. Mirum arcana illa interiora experimentis comprobari.

equal or unequal.[14] If they are equal, point *C* will be the midpoint; if unequal, one or the other of them will be greater. It seems to me I can assume straightaway that the line . . . *1 2* . . . , unbounded on both sides, can be divided in such a way that the two parts are equal, by dividing it at *B*. Then *B* will be the midpoint. But it must be seen whether this can be proved more fully. Therefore let the greater part be *C2* . . . , and the smaller *C1* . . . . I say, first, that it is not possible for one to be the triple of the other, since a line unbounded on both sides cannot be resolved into three unbounded lines, but only into two. And by the same method it can be shown that the ratio cannot be expressed by an assignable fraction of a finite number intermediate between 2 and 3, for the same reason. So the number of that fraction exceeds 2 by an infinitely small quantity. If, on the other hand, one unbounded line *C2* . . . exceeds the other *CA1* . . . by a finite quantity, such as, for example, *CB,* with this therefore assigned and definite, *B* will then be the midpoint. Hence now the lines *ED* passing through *C* will not be in the middle of the universe, but *FBG* will. In the same way, let us suppose that *B* is also the midpoint of the unbounded line *FG,* and also of the unbounded line *LH,* and of all others drawn through it. Then the point *B* will be the middle of the universe. It must be seen, however, whether it is necessary for us to be able to find a midpoint of this kind. But a difficulty seems to follow from this; let a straight line *MN* . . . be drawn unbounded towards *N;* this will be equal to *B2* . . . , since it is parallel, for it is certain that that infinite line can be carried parallel to itself in a line at right angles to itself, that is, in the line *BF.* Now it must be seen whether *BH* is equal to *B2* . . . , or unequal to it, greater or less. If *BH* is equal to it, that is, if it is the middle of the universe, as it seems possible to judge on other grounds, it must be said that *BH* will equal *MN.* Hence it is now clear how an infinite line can be moved, that is, how *BH* can be translated into *MN:* not by a continuous motion, for this is impossible, but by a leap—in other words, if it is *transcreated.*[15] For motion is nothing but transcreation.^{L6}

But there remains one great *difficulty.*[16] Let there be an infinite line—infinite either on both sides, or if you prefer, on one side—which we suppose to be actually resolved into its parts, . . . *AB, BC, CD, DE,* . . . . For there can be understood to be innumerable entities of this kind. Each of
501 these can be moved, one from *AB* into αβ, a second from *BC* into βγ, a third from *CD* into γδ, a fourth from *DE* into δε. Let us understand as many of

L6. In the margin: Perhaps it can be proved from this that every endeavour is in either perpendicular straight lines or coincident ones. And experience confirms this, as in tangents. It is surprising that these inner secrets are confirmed empirically.

etc. $\alpha \ldots \beta \ldots \chi \ldots \delta \ldots \epsilon$ *etc.*

etc. A—B—C—D—E *etc.*

etc. a . . . b . . . c . . . d . . . e *etc.*

perpendiculariter promotam. Nec hic absurdum. Jam ponatur oblique *AB, BC, CD, DE, etc.* transferri in *ab, bc, cd, de, etc.*, nonne dici poterit eodem modo totam lineam adhuc promotam. Unde sequitur magis progressam totam lineam in plagam *DE;* nam et motus eiusmodi obliquus, compositus ex perpendiculari et per sua vestigia eunte qui est si *AB* moveatur in *BC*, et *BC* in *CD*. quod fieri utique impossible est, posito corpus eundem semper occupare locum; et certe nulla erit penetratio, semper enim unum excedit altero succedente. Hinc sequitur, vel lineam interminatam esse impossibilem, vel non posse sic moveri oblique, vel non posse appellari unum totum. Sed illud hinc conclude optime posse arbitror lineam eiusmodi materialem interminatam *noninterruptam* implicare. Sed non video an interruptio nos salvet. Non enim ideo ad se propius accedent, quia uno accedente, alterum porro procedit. Revera ergo fatendum erit, Mundum esse finitum, si quantitas interminata totum sive unum est. Ita vicisset Aristoteles, et foret creaturarum quoque corporearum, numerus finitus; sed non incorporearum, ob memoriam mentium. Infinitas autem competeret Deo, ob aeternitatem, item ob alia creaturarum genera.

Una responsio adhuc supererit, scilicet quod translatio ejusmodi obliqua etiam in singulis sit impossibilis, neque a nobis perfecte intelligi, quia possibilia non possunt intelligi in singulis, nisi intellecto ordine universi, quod hic rursus illustri admodum exemplo patet. Nam alioquin ut infinitum esse posse creaturarum numerum probem, fingam similiter inde ab aeternitate, quavis hora novum positum fuisse corpus, in eadem semper recta; manifestum est, cum in singulis possibile ponam, fore tunc eius-

etc. $\alpha \ldots \beta \ldots \chi \ldots \delta \ldots \epsilon$ *etc.*

etc. A—B—C—D—E *etc.*

etc. a . . . b . . . c . . . d . . . e *etc.*

Figure 5

them as possible, taken one at a time, to have been translated one at a time. It could be said that the whole line has been moved parallel to itself and in a perpendicular direction; and this is not absurd. Now let us suppose *AB, BC, CD, DE,* . . . to be obliquely translated into *ab, bc, cd, de,* . . . : couldn't it be said that the whole line has been moved here in the same way? But from this it follows that the whole line has advanced in the direction *DE;* for an oblique motion of this kind is also one composed of a perpendicular motion and one going along its own path, i.e. one where *AB* moves into *BC*, and *BC* into *CD;* and this is not at all possible, assuming that the body always occupies the same place; and certainly there will be no penetration, for one always moves on when another comes into its place.

Hence it follows either that an unbounded line is impossible, or that it cannot be moved obliquely in this way, or that it cannot be called one whole. But I believe that what can best be concluded from this is that for an unbounded material line of this kind to be *uninterrupted by gaps* implies a contradiction. But I cannot see if an interruption of the line will save us. For the parts will not thereby get any closer together, because when one approaches, the other proceeds further ahead. Therefore it really will have to be admitted that the world is finite, if an unbounded quantity is a whole or one. In this case Aristotle would have been vindicated, and there would be a finite number of corporeal creatures too; but not of incorporeal creatures, because of the memory of minds. And yet infinity would be applicable to God, on account of eternity, as well as on account of created things of other kinds.

One response will still remain: this is that this kind of translation, oblique and also one at a time, is impossible, and not perfectly understood by us, because possibles cannot be understood one at a time without understanding the order of the universe. Again this can be made clear by a very splendid example. For otherwise, to prove that the number of created things can be infinite, I imagine similarly that at each hour from eternity onwards a new body be supposed to have come into existence, always in the same straight line; it is evident that, since it is possible to suppose them one at a time, there will then be an infinity of bodies of this kind. And so no modi infinitum. Itaque non aliter quis respondere poterit, quam negando in singulis possibile. Et si ita fingi posse videatur, quidni ergo idem respondeam statim ab initio.

502 Unum adhuc considerandum, etsi quis neget interminatam quandam esse lineam, tamen non videtur concedi posse (posito nullam esse finem materiae seu ad lineam productam semper materiam [inveniri]), quod *AB* in locum *BC,* et *BC* in locum *CD,* etc. semper simui moveri possint. Ponamus enim id fieri, et nova adhuc addatur, aliunde assumta, quae in locum ipsius *AB* primae succedat; reddita quiete, omnia erunt ut ante, et tamen facta mutatio est. Certe locum aliquem deseri, pluribus motis, successive, nec novum acquiri, impossibile. Breviter, quae sunt multa, eorum multitude, et totum et pars etc. Ergo vel negandum est infinitum actu esse possible, vel recurrendum ad nostra, quod impossibilis sit dicendus singulorum motus, etsi in ipsis per se consideratis non appareat absurditas, quia ut perfecte considerentur, Mens consideranda, quae in ipsis, et relatio facienda ad totum universum. Et ita habendum est, si in omnibus eodem modo cuncta referre possemus ad universum, appariturum nobis, quomodo revera certus tantum determinatusque rerum status sit possibilis et multa excludantur a possibilium numero in quibus nos nullam impossibilitatem invenimus, quoniam fallimur notione materiae, eamque ut extensam tantum consideramus, quod non est.

Ratio cur interminatum, seu quolibet finito maius sit aliquid, non vero infinite parvum, haec est, quod Maximum in continuo est aliquid, non vero Minimum; perfectissimum est quiddam, non vero Minimum, Deus est aliquid, nihilum non est aliquid. Totum in continuo est prius partibus. Absolutum prius limitato. Adeoque interminatum, habente terminum, cum terminus sit accessio quaedam. Nullus est numerus maximus, et nulla est linea minima.

Excutiendum adhuc, an et quatenus vera haec, v. g. quadratum est ad circulum, ut 1 ad $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$ *etc.* Nam cum dicitur *etc. in infinitum,* intelligitur ultimus numerus non esse quidem numerorum maximus, is enim nullus, sed esse tamen infinitus. Sed quoniam non determinatur quomodo? Adjiciendum enim aliquid, etiamsi numerus infinitus sumatur, ideo dicendum id non esse rigorose verum. Et quoniam circulus est nihil; utique et series ista nihil erit.

one will be able to reply in any other way than by denying it possible to suppose them one at a time. And if it seems imaginable in this way, why then should I not make the same reply at once from the start?

502 One thing still to be considered is that, even if someone should deny that a certain unbounded line exists, it still doesn't seem possible to concede (assuming there to be no end of matter, i.e. that matter can always be found for extending the line) that *AB* could always be moved into the place *BC,* and *BC* into *CD,* etc., simultaneously. For let us suppose this happens, and a new one taken from elsewhere is added, which takes the place of the first one, *AB;* when they have returned to rest, all will be as before, and yet a change will have occurred. Certainly, when many things are moved in succession, it is impossible for some place to be vacated and no new one to be acquired. And there are, in a word, many, a multiplicity of them, both whole and part, etc. Therefore either it must be denied that it is possible for an infinity to exist actually, or we must return to our previous conclusion, that the motion of the individual parts must be said to be impossible, even if there does not appear any absurdity in them considered in themselves—since in order for them to be considered perfectly, the mind which is in them must be considered, and relation must be made to the whole universe. And so it must be maintained that if in all things we could in the same way relate everything to the universe, it would be clear to us how in fact only a certain and determinate state of things is possible, and how it is that many things in which we find no impossibility are excluded from the number of possibles, since we lack a notion of matter, and consider it merely as the extended, which it is not.

The reason why the unbounded, i.e. that which is greater than anything finite, is something, and the infinitely small is not, is that in the continuum the maximum is something, and the minimum is not; the most perfect is something, the least is not, God is something, nothing is not. In the continuum, the whole is prior to its parts; the absolute is prior to the limited; and so is the unbounded prior to that which has a bound, since a bound is a kind of addition. There is no greatest number, and no least line.

We must still investigate whether and to what extent the following is true, namely, that the square is to the circle as 1 to $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$[17] For when we say 'etc.', '. . . ', or 'to infinity', the last number is not really understood to be the greatest number, for there isn't one, but it is still understood to be infinite. But seeing as the series is not bounded, how can this be the case? For something must be added, even if it is assumed to be an infinite number, so that it must be said that this is not rigorously true. And seeing as the circle is nothing, this series will of course also be nothing.

Interea superest haec difficultas. Diagonalis ad Quadratum est ratio quaedam, est enim diagonalis linea, quantitas realis, et latus itidem. Quae si numeris explicanda sit, opus etiam erit numeris infinitis. Imo in universum numeris omnibus. Dicere autem numeros omnes est nihil dicere; quare et ratio illa nihil dicit, nisi aliquid quantumvis propinquum. Non
503 ideo tamen tollitur ratio harum duarum linearum, etiamsi nulla assignetur mensura. Nisi dicas (sine Mensura), quod de angulo, id et de ratione, ipsam per se nihil esse, sed ipsum consensum divisionum; semper manentem, ut supra sinus. Imo videtur ratio semper subsistere, est enim ea ratio, per quam duae figurae sunt similes. Itaque in duabus quantitatibus similibus, id est sola magnitudine, non vero modis magnitudinis differentibus; si scilicet ipsa[e] in se sine aliis considerentur; ratio est quantitas unius determinate per relationem ad aliud. Sine similitudine non possemus intelligere rationem.

Magnitudo est rei constitution qua cognita, ipsa tota haberi potest. Videtur Totum esse etiam quod non habet partes modo habere possit. Totum est, cum ex uno fieri possunt plura. Ex uno autem fieri, est aliquid manere. Quod reapse divisum est seu Aggregatum, nescio an dici possit unum. Videtur tamen, cum sint nomina ad hoc inventa. Sed haec omnia accuratissime excutienda. Ex aliquo fieri aliud, est aliquid restare, quod ad ipsa potius quam ad aliud pertinet. Sed hoc non semper est materia. Potest esse ipsa Mens relationem quandam intelligens, ut in *transproductione,* etsi omnia nova, tamen hoc ipso quod certa lege fit haec transproductio, imitatur quodammodo motum continuum, ut polygona circulum. Et hinc dicitur unum ex alio fieri, simile quasi abusu imaginationis.

Quandocunque dicitur seriei cuiusdam infinitae numerorum dari summam, nihil aliud dici arbitror, quam seriei finitae cuiuslibet eadem regula summam dari, et semper decrescere errorem, crescente serie, ut fiat tam parvus quam velimus. Nam ipsi *per se* absolute numeri in infinitum non eunt, daretur enim numerus maximus. Sed in infinitum eunt applicati certo spatio, seu lineae interminatae in partes divisae. Hic jam ecce nova difficultas. Estne ultimus eiusmodi seriei ultimus, quae scilicet ascriberetur divisionibus lineae interminatae? non est, alioqui et ultimum in serie interminata daretur. Imo videtur dari, quia scilicet Numerus terminorum seriei, utique erit numerus ultimus, pone puncto divisionis ascribi numerum uni-

Meanwhile there remains this difficulty. Diagonal to square is a certai ratio, since the diagonal is a line, a real quantity, and the side is too. If th is to be expounded by means of numbers, there will also be a need for inf nite numbers—indeed, for all numbers in general. But to say all numbe is to say nothing; and for this reason that ratio also means nothing, unles
503 it is something as close as desired. Still the ratio of these two lines is n thereby eliminated, even if no measure is assigned. Unless (there being n measure) you also say of the ratio what you said of the angle, that in itse it is nothing but the very agreement of the divisions; an agreement that a ways remains, as did the sine above. Indeed it seems that the ratio alway subsists, since it is through this ratio that two figures are similar. And so i two similar quantities it is the only thing with magnitude, but not with di ferent modes of magnitude—that is, if they are considered in themselve without others, the ratio is the quantity of one determined by relation to th other. Without similarity, we would not be able to understand ratio.

Magnitude is that constitution of a thing by the recognition of which i can be regarded as a whole.[18] It also seems that a whole is not what ha parts, just what can have parts. A whole exists when many things can com to be out of one. But to come to be out of one is to remain something. doubt whether what is really divided, that is, an aggregate, can be calle one. It seems to be, though, because names are invented for it. But all thes matters must be investigated very carefully. For something to become an other thing is for something to remain which pertains to it rather than to th other thing. But this is not always matter. It can be mind itself, understand ing a certain relation: for instance, in *transproduction,* even though every thing is new, still, by the very fact that this transproduction happens by certain law, continuous motion is imitated in a way, just as polygons imi tate the circle. And hence one may be said to come out of the other, by similar abuse, as it were, of the imagination.

Whenever it is said that a certain infinite series of numbers has a sum, am of the opinion that all that is being said is that any finite series with th same rule has a sum, and that the error always diminishes as the serie increases, so that it becomes as small as we would like.[19] For numbers d not *in themselves* go absolutely to infinity, since then there would be greatest number. But they do go to infinity when applied to a certain spac or unbounded line divided into parts. Now here there is a new difficulty. I the last number of a series of this kind the last one that would be ascribe to the divisions of an unbounded line? It is not, otherwise there would als be a last number in the unbounded series. Yet there does seem to be, because the number of terms of the series will be the last number. Suppose t the point of division we ascribe a number always greater by unity than th

tate majorem semper praecedente, utique Numerus terminorum erit ultimus seriei, at vero non datur seriei ultimus, quia est interminata; inprimis si series sit utrinque interminata; ergo concludemus tandem quod nulla sit multitudo infinita. Unde sequeretur nec infinitas res esse. Vel dicendum infinitas res non esse unum totum, seu non esse earum aggregatum. Si res
504 infinitae esse non possent, foret mundus necessario tempore et loco finitus, sed mundum tempore finitum esse, non videtur possibile. imo sequeretur, et aliquando res cessaturas, omniaque in nihilum reditura, nam alioqui omnia futura essent infinita. Itaque si dicas in [serie] interminata non dari ultimum finitum numerum inscriptibilem, posse tamen infinitum dari: Respondeo, ne hunc quidem dari posse, si nullum sit ultimum. Ad hanc ratiocinationem non aliud habeo quod respondeam, quam numerum Terminorum non semper esse ultimum seriei. Patet scilicet etsi in infinitum augeantur numeri finiti, nunquam nisi finita aeternitate, id est nunquam, pervenire ad infinitos. Subtilis admodum haec consideratio est.

Communicata ex literis Domini Schulleri — Aiii19

275 COMMUNICATA EX LITERIS D. SCHULL.

(1) *Demonstrat quod omnis substantia sit infinita, indivisibilis et unica.*
Per substantiam intelligit id quod in se est, et per se concipitur, hoc est id cuius Idea vel conceptus ex Idea vel conceptu alterius rei non oritur.[L1]

L1. Videntur a nobis *per se concipi*, quorum termini sive Voces sunt indefinibiles, seu quorum ideae irresolubiles, ut Existentia, Ego, perceptio idem, mutatio; tum qualitates sensibiles; ut calor, frigus, lumen, etc. *Per se vero intelligi* non nisi id cuius omnia requisita concipimus, sine alterius rei conceptu, sive id quod sibi ipsi existendi ratio est. *Intelligere* enim nos vulgo res dicimus cum eorum generationem concipimus, sive modum quo producuntur. Unde per se intelligitur, id tantum quod causa sui est, sive quod necessarium est, sive Ens a se. Adeoque id hinc concludi potest, nos, si Ens necessarium intelligamus, id per se intellecturos. Dubitari vero potest, an Ens necessarium a nobis intelligatur, imo an possit intelligi, etsi possit sciri sive cognosci.

Conceptus distinguit in claros tantum, et claros distinctosque simul. Omnis con-

preceding one, then of course the number of terms will be the last number of the series. But in fact there is no last number of the series, since it is unbounded; especially if the series is unbounded at both ends. Therefore we conclude finally that there is no infinite multiplicity, from which it will follow that there is not an infinity of things, either. Or it must be said that an infinity of things is not one whole, i.e. that there is no aggregate of them. If an infinity of things could not exist, the world would be necessar-
504 ily finite in time and place, but for the world to be finite in time does not seem possible. Indeed, it would then follow also that at some time things will come to an end, and everything will be reduced to nothing, for otherwise [the number of] all future things would be infinite. Thus if you say that in an unbounded [series][20] there exists no last finite number that can be written in, although there can exist an infinite one: I reply, not even this can exist, if there is no last number. The only other thing I would consider replying to this reasoning is that the number of terms is not always the last number of the series. That is, it is clear that even if finite numbers are increased to infinity, they never—unless eternity is finite, i.e. never—reach infinity. This consideration is extremely subtle.

17. Annotated Excerpts from Spinoza[1] — Aiii19

[2nd half of April 1676?][2]

275 ITEMS COMMUNICATED IN MR. SCHULLER'S LETTER[3]

(1) *He demonstrates that every substance is infinite, indivisible, and unique.*[4]
By substance he understands that which is in itself, and is conceived through itself,[L1] that is, that whose idea or concept does not arise from the idea or concept of another thing.[5]

L1. It seems that we *conceive through themselves* those things whose terms or expressions are undefinable, i.e. whose ideas are irresolvable, such as existence, the ego, perception, the same, change; as well as sensible qualities, such as heat, cold, light, etc. But something is *understood through itself* only if we conceive all its requisites without having conceived another thing, i.e. only if it is the reason for its own existence. For we commonly say that we *understand* things when we can *conceive* their generation, i.e. the way in which they were produced. Hence we understand through itself only that which is its own cause, i.e. that which is necessary, i.e. is a being in itself. And so it can be concluded from this that if we understood a necessary being, we would understand it through itself. But it can be doubted whether we do understand a necessary being, or, indeed, whether it could be understood even if it were known or recognized.

He distinguishes concepts into the merely clear, and the clear and distinct to-