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THE HYPERREAL LINE

1. INTRODUCTION

The aim of this article is to explain that the hyperreal line is, what it looks like, and what it is good for. Near the beginning of the article we shall draw pictures of the hyperreal line and sketch its construction as an ultrapower of the real line. In the middle part of the article, we shall survey mathematical results about the structure of the hyperreal line. Near the end, we shall discuss philosophical issues concerning the nature and significance of the hyperreal line.

The hyperreal number system is a very rich extension of the real number system which preserves all first-order properties. It contains infinite and infinitesimal numbers. Another important feature is the 'hyperfinite grid', which is an infinite set of equally spaced points on the unit interval which has the first-order properties possessed by all sufficiently large finite grids.

We begin with a brief history of the hyperreal line. Archimedes (287–212 B.C.) in his manuscript 'The Method' introduced infinitesimals as a method for the discovery of mathematical results. The basic rules of the calculus were discovered in the 1500s and 1600s by reasoning informally with infinitesimals. The method remained dominant until the middle of the seventeenth century, and was the motivation for Leibniz' differential notation dx. Leibniz (1646–1716) correctly anticipated the modern viewpoint; he regarded the infinitesimals as ideal numbers like the imaginary numbers, and proposed his law of continuity: "In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the terminus may also be included." This 'law' is far too imprecise by present standards, but was a forerunner of the modern Transfer Principle that the hyperreal number system has the same first-order properties as the real number system.

Bishop Berkeley effectively criticized the logical inconsistencies in the intuitive use of infinitesimals in 1734. A precise treatment of either the real or the hyperreal number system was beyond the state of the art at the time. In the 1870s set theory was developed by Cantor, Bolzano,

P. Ehrlich (ed.), Real Numbers, Generalizations of the Reals, and Theories of Continua, 207–237.

Dedekind, and Weierstrass to the point where the real line could be precisely defined, and Weierstrass based the classical rigorous treatment of the calculus on the real line.

A mathematically precise construction of a hyperreal line had to wait for developments in the area of mathematical logic, in particular model theory. To construct the hyperreal number system, one needs either the compactness theorem or some version of the ultrapower construction. In 1927, Thoralf Skolem introduced the ultrapower construction to obtain a nonstandard model of the first-order theory of arithmetic. The compactness theorem for countable languages was proved by Gödel in 1930, and the general case of the compactness theorem was proved by Malcev in 1936. In 1948, Hewitt used the ultrapower construction, from the point of view of algebra rather than logic, to obtain a real closed extension of the ordered field of real numbers, and introduced the name hyperreal numbers. In 1955, Łoś introduced the ultrapower construction in general and proved the Transfer Principle that the ultrapower preserves all first-order properties. The beginnings of an infinitesimal approach to analysis was developed in 1958 by Laugwitz and Schmieden (Laugwitz and Schmieden, 1958). The big step came in 1960, when Abraham Robinson realized that the Transfer Principle could be used to give a mathematically correct development of all of analysis based on the hyperreal number system, and gave the method the name nonstandard analysis. Robinson's early papers and his book (Robinson, 1966) developed the hyperreal number system within the theory of types. The ultrapower treatment was popularized in the monograph by Luxemburg (1962). Since then, it has become evident that the hyperreal number system is a tool which is broadly applicable in many areas of pure and applied mathematics.

For the last century, the mathematical community has had a strong aversion to the notion of an infinitesimal. The original reason for this aversion is the historical fact that infinitesimals were used incorrectly in the development of the calculus, and the errors were corrected by the treatment of Weierstrass based on the real line (banishing infinitesimals). More recently, the influence of the hyperreal line on mathematics has been held back by the fact that mathematical logic has seldom been regarded as an essential part of the graduate mathematics curriculum. Although it is formally possible to develop the hyperreal number system without mathematical logic, one must be comfortable with mathematical logic in order to make effective use of the hyperreal numbers as a research tool.

Some recent books on the hyperreal line and its role in mathematics are (Albeverio *et al.*, 1986; Cutland, 1988; Hurd and Loeb, 1985; Robert, 1988; Strogan and Bayod, 1986). For an elementary treatment at the freshman calculus level, see (Keisler, 1986). For background material and references in the theory of models, see (Chang and Keisler, 1990).

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2. INFINITESIMALS

One of the most striking and best known features of the hyperreal line is the presence of infinitesimals and infinite numbers. However, infinitesimals are only part of the story. They arise not only in the hyperreal line but in much simpler structures, namely, in any ordered field which does not have the Archimedean property.

In the beginning stages of Robinson's analysis, only a few algebraic rules are needed, and one can go surprisingly far into the infinitesimal calculus of algebraic functions with such rules. More power is not required until one gets to the transcendental functions. These rules are contained in the notion of a non-Archimedean real closed ordered field. In this section and the next shall discuss the picture of the line from a non-Archimedean viewpoint. We begin with the basic definitions.

DEFINITION. Given an ordered field $\mathbb{F} = \langle F, +, \cdot, 0, 1, \leq \rangle$, we shall identify each natural number $n \in \mathbb{N}$ with the field element $1 + \ldots + 1$ (n times). An element x of $\mathbb{F}$ is said to be *infinitesimal* iff $|x| \leq 1/n$ for all positive integers n, *finite* iff $|x| \leq n$ for some integer n, and *infinite* iff $|x| \geq n$ for all integers n. Two elements x, y are said to be *infinitesimally close*, in symbols $x \approx y$, iff $y = x + d$ for some infinitesimal d.

DEFINITION. An ordered field is said to be *non-Archimedean* iff it has at least one positive infinite element, and *Archimedean* otherwise.

A fundamental classical result is that an ordered field is Archimedean if and only if it is isomorphic to an ordered subfield of the field $\mathbb{R}$ of real numbers. For example, the ordered fields of real numbers, rational numbers, and real algebraic numbers are Archimedean.

Examples. Here are some examples of non-Archimedean ordered fields which are defined without recourse to methods from mathematical logic. Each example except (c) is an extension of the field of real numbers.

(a) The field $\mathbb{R}(x)$ of all rational functions over $\mathbb{R}$ of one variable x (i.e. quotients of polynomials in x with real coefficients), with the ordering determined by stipulating that x is positive infinite.

(b) The real closure of $\mathbb{R}(x)$, that is, the smallest ordered field containing $\mathbb{R}(x)$ which is closed under taking roots of polynomials of odd degree and square roots of positive elements.

(c) The field $\mathbb{Q}(x_1, \ldots, x_n)$ of rational functions over the rationals $\mathbb{Q}$ in n variables, with the ordering determined by making x_1 positive infinite and x_{i+1} greater than any power of x_i for $i = 1, \ldots, n-1$.

(d) The field of Laurent series, i.e. formal power series of the form $\sum_{-k}^{\infty} a_n x^n$ with real coefficients and $k \in \mathbb{N}$, making x positive infinite.

(e) The ordered field of surreal numbers of Conway [*Co*].

Another example of a non-Archimedean ordered field is the ordered field of hyperreal numbers, which will be introduced in due course.

BLANKET ASSUMPTION. Throughout this and the next section, $\mathbb{F}$ will denote a non-Archimedean ordered field which contains the real number field $\mathbb{R}$.

PROPOSITION 1. (Algebra of infinitesimals). (i) *The finite elements of $\mathbb{F}$ form a convex subring. That is, sums, differences, and products of finite elements are finite, and any element between two finite elements is finite.*

(ii) *The infinitesimal elements of $\mathbb{F}$ form a convex ideal in the ring of finite elements. That is, sums and differences of infinitesimals are infinitesimal, the product of an infinitesimal and a finite element is infinitesimal, and any element between two infinitesimals is infinitesimal.*

(iii) *If H is positive infinite, then each $K \geq H$ is positive infinite, $H + x$ is positive infinite for all finite x, and $H \cdot y$ is positive infinite for all positive noninfinitesimal y.*

(iv) *If x is nonzero infinitesimal then $1/x$ is infinite, and if y is infinite then $1/y$ is infinitesimal.*

We can see now that a non-Archimedean ordered field has infinitely many different infinitesimals and infinite elements. By definition, there is at least one positive infinite element H. By (iii) there are infinitely many different positive infinite elements. By (iv), $d = 1/H$ is positive infinitesimal. By (i), all finite multiple of d, all finite powers d^n of d, etc. are infinitesimal. Given any positive infinitesimal c, $-c$ is a negative infinitesimal.

The set of infinitesimals may be thought of as a small cloud centered at zero, and the set of finite elements a large cloud centered at zero. About each $x \in \mathbb{F}$, there is another small cloud, called the *monad* of x, consisting of the set of all elements which are infinitesimally close to x, and a large cloud, the *galaxy* of x, consisting of the set of all elements which are at a finite distance from x. The monad of zero is the set of infinitesimals, and the galaxy of zero is the set of finite elements. Note that each galaxy is a union of monads.

3. STANDARD PARTS AND ELEMENTARY EXTENSIONS

We shall now look at the real number field $\mathbb{R}$ and a non-Archimedean extension $\mathbb{F}$ as a pair of structures. We introduce the standard part function, which associates a real number with each finite element of $\mathbb{F}$.

DEFINITION. Let x be a finite element of $\mathbb{F}$. A real number r is said to be the *standard part* of x, in symbols $r = \mathrm{st}(x)$, if r is infinitesimally close to x. Infinite elements do not have standard parts.

PROPOSITIONS 2. *Each finite element of $\mathbb{F}$ has a unique standard part.*

Hint. Show that if $x \in F$ is finite, then the least upper bound of the set of all real $r < x$ is the standard part of x.

Propositions 1 and 2 provide us with the picture, shown in Figure 1, of a non-Archimedean line floating above the real line.

In the normal real scale without magnification, the non-Archimedean line looks like the real line. But if we train an 'infinitesimal microscope' on a finite element x, and set the magnification to be an infinite factor y, the monad of x will be infinitely magnified and points which

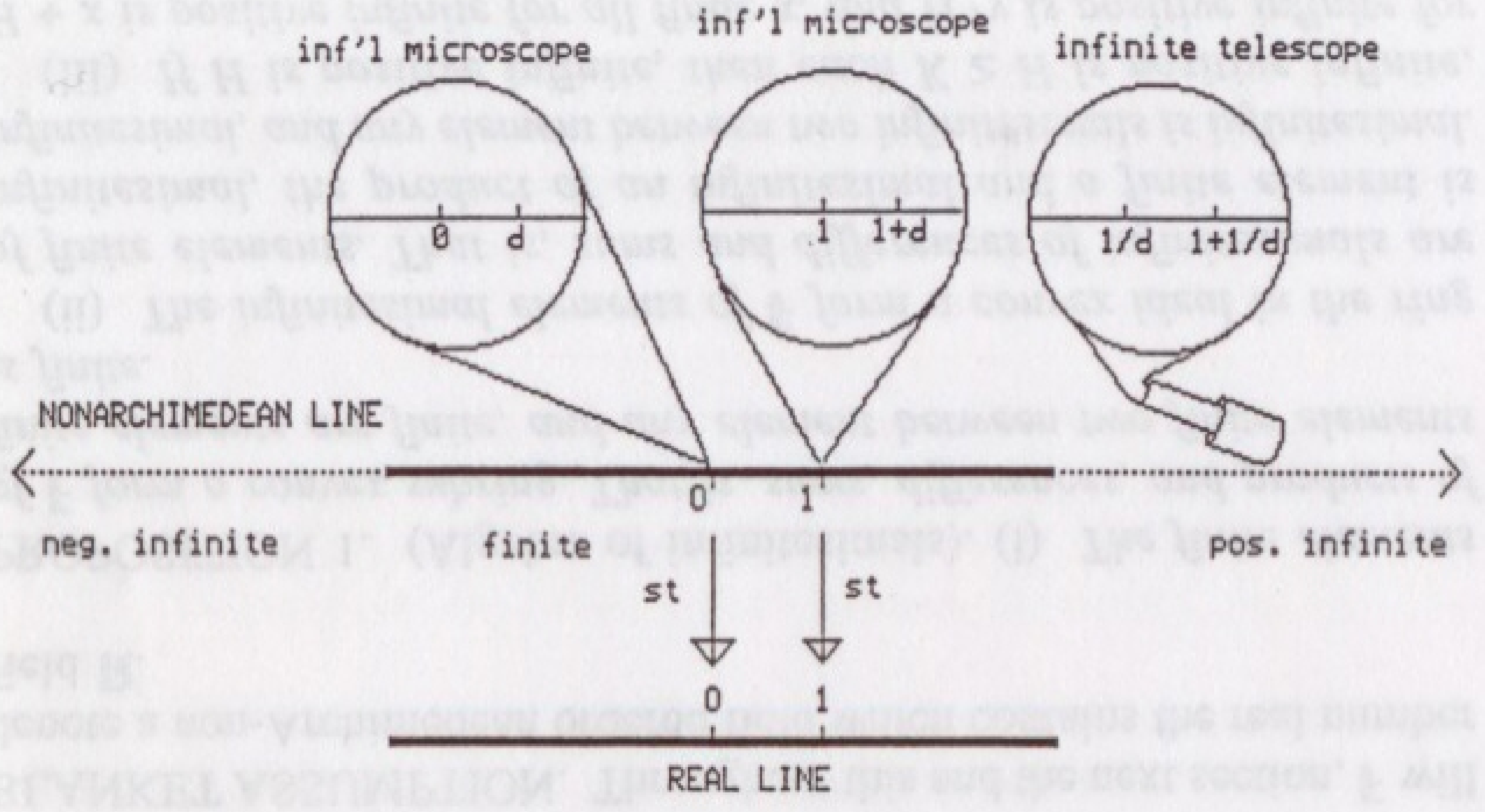

Fig. 1. The real and non-Archimedean Lines.

are at the infinitesimal distance $1/y$ apart will be distinguishable. If we aim an 'infinite telescope' at a positive infinite element, we will see positive infinite elements in the normal real scale.

An ordered field is said to be *real closed* iff every positive element has a square root and every polynomial of odd degree has a root. Among the examples in the preceding section, (b) and (e) are real closed ordered fields. The ordered fields of real numbers, real algebraic numbers, and hyperreal numbers are also real closed. In the classical paper (Tarski and McKinsey, 1948), Tarski proved that all real closed ordered fields have the same properties which are expressible in first-order logic. More precisely:

PROPOSITION 3. (i) *The first-order theory of real closed ordered fields is complete, that is, all real closed ordered fields satisfy the same sentences of first-order logic.*

(ii) *The first-order theory of real closed ordered fields is model complete, that is if* $\mathbb{G} \subseteq \mathbb{F}$ *are real closed ordered fields, then any formula of first-order logic with names for elements of* $\mathbb{G}$ *is true in* $\mathbb{F}$ *if and only if it is true in* $\mathbb{G}$.

In particular, if $\mathbb{F}$ is a real closed ordered field which contains $\mathbb{R}$, then any first-order formula with names for real numbers is true in $\mathbb{F}$ if and only if it is true in the real numbers. In model theory, this is expressed briefly by saying that $\mathbb{F}$ is an *elementary extension* of $\mathbb{R}$.

There is an important loophole in the result of Tarski. It applies only to formulas in the language of ordered fields, that is, formulas built up from the predicates =, ≤, the function symbols +, ·, and the constants 0 and 1. Nothing has been said about how one might extend other real relations or functions to the larger set $\mathbb{F}$. For example, the exponential functions, the trigonometric functions, and the set of natural numbers cannot be defined in the language of ordered fields. One of the main properties of the hyperreal line will be the Transfer Principle, which says that the hyperreal number system is an elementary extension of the real number system even in the full language which has a symbol for every relation and function over $\mathbb{R}$.

4. THE HYPERREAL NUMBERS

The hyperreal number system can be introduced either by an axiomatic approach which lists its properties and proves that a structure with those properties exists, or by an explicit construction. In this article we shall construct the hyperreal number system explicitly as an ultrapower of the real number system. This has the advantage of giving the reader a concrete structure, but the disadvantage of being less natural than the axiomatic approach. Moreover, the ultrapower construction is special, in the sense that not all hyperreal number systems allowed by the axiomatic approaches are ultrapowers of the real number system. In Section 12 we shall touch on two axiomatic approaches which extend not only the real line but the whole set theoretical universe in which the real line lives.

We now define the notion of an ultrapower of the real number system $\mathbb{R}$. Let I be an infinite set, the *index set*. Let us consider a family U of subsets of I. An *ultrafilter over* I is a set U of subsets of I such that

(1) If $X \in U$ and $X \subseteq Y \subseteq I$, then $Y \in U$;
(2) If $X, Y \in U$ then $X \cap Y \in U$;
(3) For all $X \subseteq I$, exactly one of the sets X, $I \backslash X$ belongs to U.

Given an ultrafilter U, we shall form a new structure $^*\mathbb{R}$, called the *ultrapower* of $\mathbb{R}$ modulo U. In order to see how the conditions (1)–(3) arise naturally, let us temporarily consider an arbitrary set U of subsets of I. We begin with the set $\mathbb{R}^I$ of all functions from I into $\mathbb{R}$. Our plan is to identify certain elements of $\mathbb{R}^I$ with each other, so the elements of the ultrapower will be equivalence classes of elements of $\mathbb{R}^I$. We first deal with the properties of equality and order, and introduce two relations $=_U$ and $\le_U$ on $\mathbb{R}^I$ as follows:

$f =_U g$ iff $\{i \in I : f(i) = g(i)\} \in U$,
$f \le_U g$ iff $\{i \in I : f(i) \le g(i)\} \in U$,

The following proposition motivates the definition of an ultrafilter.

PROPOSITION 4. *Let U be a family of subsets of I. Then U is an ultrafilter over I if and only if:*
(a) *$=_U$ is an equivalence relation on $\mathbb{R}^I$ with more than one class;*
(b) *If $f =_U g$ then $f \le_U g$;*
(c) *For all $f, g \in \mathbb{R}^I$, either $f \le_U g$ or $g \le_U f$.*

Hint. Use (a) to prove (2) in the definition of an ultrafilter, use (b) to prove (1), and then use (a) and (c) to prove (3). The other direction is easy.

Hereafter we assume that U is an ultrafilter over I. For $f \in \mathbb{R}^I$, we let f_U be the $=_U$ equivalence class of f, and define the *ultrapower* of $\mathbb{R}$ modulo U to be the set $^*\mathbb{R}$ of all equivalence classes,

$$^*\mathbb{R} = \{f_U : f \in \mathbb{R}^I\}.$$

We identify each real number $r \in \mathbb{R}$ by the equivalence of its constant function, $r = (I \times \{r\})_U$, so that $\mathbb{R} \subseteq {}^*\mathbb{R}$. If $\mathbb{R}$ is a proper subset of $^*\mathbb{R}$, we call the ultrapower *nontrivial*. The following proposition, which depends on the axiom of choice, insures the existence of nontrivial ultrapowers.

PROPOSITION 5. (i) (Tarski, 1930). *Every infinite set has an ultrafilter which is not closed under countable intersections.*

(ii) *The ultrapower of $\mathbb{R}$ modulo U is trivial if and only if U is closed under countable intersections.*

We now define relations and operations on $^*\mathbb{R}$, by generalizing the definition of $=_U$ and $\le_U$. Let $\mathcal{F}$ be the set of all relations and functions on $\mathbb{R}$. By the *full structure over* $\mathbb{R}$ we mean the model $\langle \mathbb{R}, S : S \in \mathcal{F}\rangle$. We shall expand $^*\mathbb{R}$ to a model for the vocabulary $\mathcal{F}$. Let S be an n-ary relation and G an n-ary function on $\mathbb{R}$. For all $f_1, \ldots, f_n \in \mathbb{R}^I$, we define

$$^*S(f_{1U}, \ldots, f_{nU}) \text{ iff } \{i \in I : S(f_1(i), \ldots, f_n(i)) \in U\},$$
$$^*G(f_{1U}, \ldots, f_{nU}) = \langle G(f_1(i), \ldots, f_n(i)) : i \in I\rangle_U.$$

That is, a relation holds in the ultrapower iff it holds for U– almost all i, and a function in the ultrapower is defined component-wise. It can be checked that these are proper definitions, that is, they depend only on the equivalence classes of $f_1, \ldots, f_n$ and not on the functions themselves. We are now ready to state our definition of a hyperreal number system.

DEFINITION. By a *hyperreal number system* we shall mean a structure of the form $\langle {}^*\mathbb{R}, {}^*S : S \in \mathcal{F}\rangle$ where $^*\mathbb{R}$ is a nontrivial ultrapower of the real line $\mathbb{R}$. By a *hyperreal line* we mean the ordered set $\langle {}^*\mathbb{R}, {}^*\le\rangle$ in a hyperreal number system.

We shall assume hereafter that $^*\mathbb{R}$ is a nontrivial ultrapower of $\mathbb{R}$. Relations and functions over $\mathbb{R}$ will be called *real*, and stars of real relations and functions will be called *standard*. Thus a relation T over $^*\mathbb{R}$ is standard iff $T = {}^*S$ for some real relation S. For example, the set $\mathbb{N}$ of natural numbers is a real set, and the set $^*\mathbb{N}$ of *hypernatural numbers* is a standard set.

The usefulness of the hyperreal number system is based on two fundamental properties, the Transfer and Saturation Principles. The Transfer Principle formalizes the intuitive idea that the hyperreal number system should be as much like the real number system as possible. The Saturation Principle, stated in the next section, captures the fact that the hyperreal number system is very rich.

TRANSFER PRINCIPLE (Łoś 1955). Any hyperreal number system $\langle {}^*\mathbb{R}, {}^*S : S \in \mathcal{F}\rangle$ is an elementary extension of the full structure $\langle \mathbb{R}, S : S \in \mathcal{F}\rangle$. That is, every n-tuple of real numbers satisfies the same first-order formulas in $\langle \mathbb{R}, S : S \in \mathcal{F}\rangle$ as it satisfies in $\langle {}^*\mathbb{R}, {}^*S : S \in \mathcal{F}\rangle$.

The proof is by induction on the complexity of formulas.

It follows from the Transfer Principle and the fact that $^*\mathbb{R}$ is a proper extension of $\mathbb{R}$ that $^*\mathbb{R}$ is a non-Archimedean real closed ordered field. Moreover, all other real functions, such as the exponential and trigonometric functions, satisfy the same inequalities in the hyperreal number system that they satisfy in the real number system.

5. INTERNAL SETS AND SATURATION

The concept of an internal set is of overriding importance in Robinson's analysis. Intuitively, the internal sets are those sets of hyperreal numbers which inherit the first-order properties of sets of the real numbers. Actually, we work with n-ary relations rather than only sets. Ordinarily, the internal relations are defined as those relations which are ultraproducts of real relations. We shall give a simpler but slightly stronger definition here which will be sufficient for our purposes; our internal relations will just be the sections of standard relations. We use the vector notation **a** for an n-tuple $\langle a_1, \ldots, a_n \rangle$.

DEFINITION. An n-ary relation T on $^*\mathbb{R}$ is said to be *internal* iff there is a real $(n+1)$-ary relation S and a hyperreal number b such that

$$T = \{\mathbf{a} \in {}^*\mathbb{R}^n : {}^*S(\mathbf{a}, b)\}.$$

Remark. It can be shown that any relation which is internal in the above sense is an ultraproduct modulo U of a family of standard relations. The converse also holds if the index set I has cardinality at most 2^ω.

The next two results are useful consequences of the Transfer Principle.

THE OVERSPILL PRINCIPLE. Every nonempty bounded internal subset of $^*\mathbb{R}$ has a supremum in $^*\mathbb{R}$. *Hint.* Apply Transfer to the sentence which states that every nonempty bounded section of a relation has a supremum.

Examples. By the Overspill Principle, The following subsets of $^*\mathbb{R}$ are not internal, since they are nonempty and bounded but have no supremum: $\mathbb{N}$, $\mathbb{R}$, the monad of 0, the galaxy of 0. As these examples show, the hyperreal line is not completely ordered. This example contains a warning: the Transfer Principle does not hold for second-order formulas.

The Overspill Principle is used in almost every application of the hyperreal number system. A typical exercise: Use the Overspill Principle to show that for each real function f, $\lim_{x\to\infty} f(x) = 0$ if and only if $^*f(x) \approx 0$ for all positive infinite $x \in {}^*\mathbb{R}$.

In order to make effective use of the hyperreal number system, it is vitally important to be able to recognize an internal relation. In practice, we usually use the following principle to show that a relation is internal. It is a corollary of the Transfer Principle.

INTERNAL DEFINITION PRINCIPLE. Let $\phi(\mathbf{x}, \mathbf{y})$ be a formula in the vocabulary $\mathcal{F}$, and let **b** be a tuple of hyperreal numbers. Then the relation

$$\{\mathbf{a} \in {}^*\mathbb{R}^m : \phi(\mathbf{a}, \mathbf{b}) \text{ holds in } \langle {}^*\mathbb{R}, {}^*S : S \in \mathcal{F}\rangle\}$$

is internal.

Example. (a) Any standard relation $^*S$ over $^*\mathbb{R}$ is internal.

(b) The collection of internal sets is closed under finite unions, finite intersections, and complements.

(c) If F is an internal function, then the domain and range of F are internal.

(d) For each pair of hyperreal numbers c, d, the *closed interval

$$[c, d] = \{a \in {}^*\mathbb{R} : c \,{}^*\!\le a \text{ and } a \,{}^*\!\le d\}$$

is internal.

(e) For each hypernatural number $H \in {}^*\mathbb{N}$, the set

$$\{0, \ldots, H-1\} = \{K \in {}^*\mathbb{N} : K \,{}^*\!< H\}$$

is internal.

We now state the second basic property of the hyperreal number system.

SATURATION PRINCIPLE (Keisler, 1964). Any countable decreasing chain of nonempty internal subsets of $^*\mathbb{R}$ has a nonempty intersection.

Hint. Use Proposition 5 to get a decreasing chain I_n of sets in the ultrafilter U with empty intersection. Given a chain S_n of nonempty internal sets, choose $f(i)$ so that whenever $i \in I_n$, $f(i)$ belongs to as many of the sets $S_{i0}, \ldots, S_{in}$ as possible.

Examples. Here are some easy consequences of the Saturation Principle which add to our picture of the hyperreal line.

(a) No countable infinite set is internal.

(b) No countable strictly increasing sequence of hyperreal numbers has a least upper bound.

(c) Any countable set of infinitesimals has an infinitesimal upper bound. Any countable set of hyperreal numbers has a hyperreal upper bound.

6. THE HYPERFINITE GRID

We now introduce the hyperfinite sets. Intuitively, a hyperfinite set is a subset of $^*\mathbb{R}$ which inherits the first order properties of finite subsets of $\mathbb{R}$. By analogy with the internal sets, a hyperfinite set is ordinarily defined as an ultraproduct of a family of finite sets. As in the case of internal sets, we shall use a simpler but slightly stronger definition; our hyperfinite sets will be the sections of standard relations $^*S$ such that every section of the original real relation S is finite.

DEFINITION. A subset T of $^*\mathbb{R}$ is said to be *hyperfinite* iff there is a real relation S such that

$$\{a \in \mathbb{R} : S(a, b)\} \text{ is finite for all } b \in \mathbb{R}$$

and

$$T = \{a \in {}^*\mathbb{R} : {}^*S(a, b)\} \text{ for some } b \in {}^*\mathbb{R}.$$

Clearly, each hyperfinite set is internal, and each finite subset of $^*\mathbb{R}$ is hyperfinite. An example of a hyperfinite set which is infinite is the set $\{0, \ldots, H-1\}$ where H is a positive infinite hypernatural number. By the Transfer Principle, a set T is hyperfinite if and only if there exists a hypernatural number H and an internal bijection F from $\{0, \ldots, H-1\}$ onto T; we call H the *internal size* of T. Moreover, any internal subset of a hyperfinite set is hyperfinite.

By writing integers in base 2 and using Transfer, a hyperfinite set $X \subseteq {}^*\mathbb{N}$ may be coded by the hypernatural number $c(X) = \sum\{2^J : J \in X\}$. This coding reduces properties of hyperfinite subsets of $^*\mathbb{N}$ to properties of elements of $^*\mathbb{N}$, and is useful in combination with the Internal Definition Principle.

PROPOSITION 6. *Here are some consequences of the Transfer and Saturation Principles which help to give us a picture of the hyperfinite sets.*

(i) *Any nonempty hyperfinite subset of $^*\mathbb{N}$ has a greatest element.*

(ii) *Given an increasing chain S_n and a decreasing chain T_n of hyperfinite sets such that $S_n \subseteq T_n$ for all $n \in \mathbb{N}$, there exists a hyperfinite set X such that $S_n \subseteq X \subseteq T_n$ for all $n \in \mathbb{N}$.* *Hint.* Use part (ii) and Saturation.

(iii) *The union of a countable strictly increasing chain of hyperfinite sets is external.* *Hint.* Similar to (ii).

(iv) *Any function $f : \mathbb{N} \to {}^*\mathbb{N}$ can be extended to an internal function $F : {}^*\mathbb{N} \to {}^*\mathbb{N}$.* *Hint.* Use part (ii), Saturation, and the coding $c(X)$.

One of the most important features of the hyperreal line is that each interval can be partitioned into hyperfinitely many subintervals of the same length. This feature is captured by the notion of a hyperfinite grid.

DEFINITION. Choose a positive infinite hypernatural number H. By the *hyperfinite grid* with mesh $1/H$ we mean the hyperfinite set of all multiples of $1/H$ between $-H$ and H,

$$\mathbb{H} = \{K/H : K \in {}^*\mathbb{Z} \text{ and } |K| \leq H^2\}.$$

We think of $\mathbb{H}$ as the set

$$\mathbb{H} = \left\{ -H, -H + \frac{1}{H}, -H + \frac{2}{H}, \ldots, -\frac{2}{H}, -\frac{1}{H}, \right.$$
$$\left. 0, \frac{1}{H}, \frac{2}{H}, \ldots, H - \frac{2}{H}, H - \frac{1}{H}, H \right\}$$

We usually take H so that every standard natural number divides H. It then follows that each standard rational number belongs to $\mathbb{H}$, that is, $\mathbb{Q} \subseteq \mathbb{H}$.

The notions of an interval, monad, and galaxy in $\mathbb{H}$ are defined by restricting the original notions to $\mathbb{H}$ in the obvious way. The hyperfinite grid is not a field, and in fact is not even closed under addition, subtraction, multiplication, or division. The coarse picture of the hyperfinite grid as an ordered set is much like the hyperreal line or non-Archimedean line, with a monad surrounding each real number, a finite part, and a negative and positive infinite part. However, the finer details, shown in Figure 2, are markedly different.

A hyperfinite grid $\mathbb{H}$ has a least and a greatest element and is *discretely ordered*, that is, each element except the least has an immediate

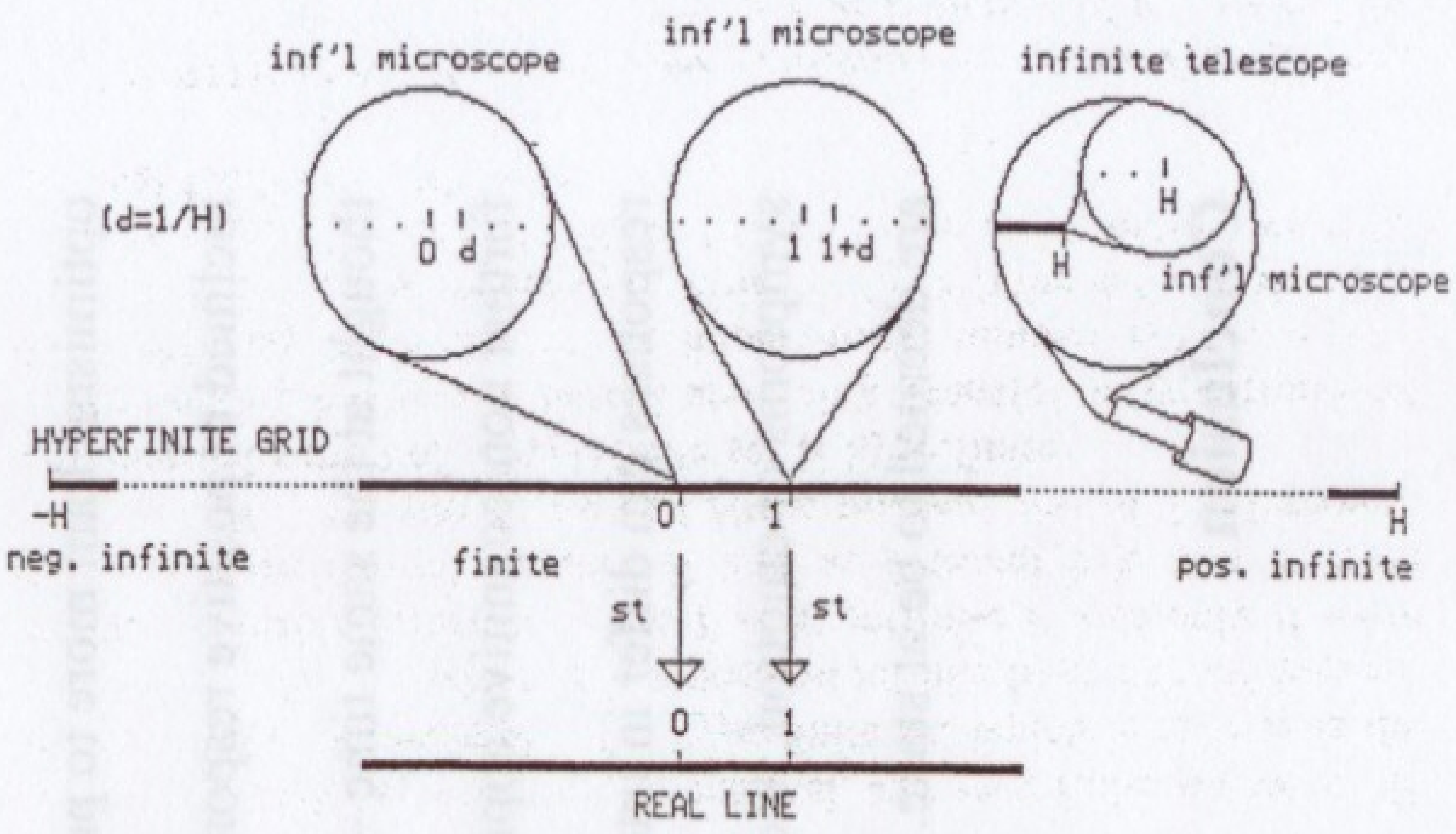

Fig. 2. The real line and the hyperfinite grid.

predecessor and each element except the greatest has an immediate successor. If we train an infinitesimal microscope on an element of the hyperfinite grid and set the magnification to H, the portion of the grid inside the microscope will look like a portion of the set $\mathbb{Z}$ of integers. If we train a microscope on the greatest element H and aim the telescope at this microscope, we will see the upper end of the hyperfinite grid, which will look like a portion of the set of negative integers with 0 at H.

The hyperfinite grid, like any discretely ordered set with a first and last element, many be divided into a first block which is a copy of the positive integers, a linearly ordered set of blocks which are copies of the set $\mathbb{Z}$ of integers, and a last block which is a copy of the negative integers. Moreover, the ordering of the blocks is dense, because if x and y are in different blocks, then $1/2(x + y)$ is in a block strictly between the two.

The set of real numbers $\mathbb{R}$ is not contained in $\mathbb{H}$. In fact, $\mathbb{H}$ is a subset of the set $^*\mathbb{Q}$ of hyperrational numbers, and no irrational number belongs to $^*\mathbb{Q}$. On the other hand, every real number is infinitely close to some element of $\mathbb{H}$, so the standard part function maps the finite part of $\mathbb{H}$ onto $\mathbb{R}$. *Hint.* Let r be real. By Transfer, there is a greatest element $[r]$ of $\mathbb{H}$ such that $[r] \leq r$, and $[r]$ is infinitely close to r.

7. THE LOEB MEASURE

One of the benefits of the hyperfinite grid is that one can calculate with it as if it were finite. Such calculations are legal only for internal objects. The internal subsets of the hyperfinite grid are the 'simplest' subsets; they are themselves hyperfinite, and play a role somewhat like the intervals, or finite unions of intervals, on the real line. A common theme is to approximate a set in which one is interested by an internal set.

In this section we illustrate this theme with the Loeb measure construction, which has proved to be an extremely powerful tool. The idea is to assign an equal infinitesimal weight $1/H$ to each element of the hyperfinite grid, compute an *internal measure* which assigns a hyperreal value to each internal subset by adding up weights, take standard parts to get a real valued measure, and then use Saturation to extend to a countably additive measure in the classical sense. We choose the weight $1/H$ because then for any $a, b \in \mathbb{H}$, the internal measure of an interval $\mathbb{H} \cap [a, b)$ will be equal to the length $b - a$. Thus the internal measure of an interval of infinite length will be infinite. To avoid side issues involving sets of infinite measure, from now on we shall concentrate on the hyperfinite unit interval $\mathbb{H}_1$ consisting of all $x \in \mathbb{H}$ such that $0 \leq x < 1$.

DEFINITION. Let A be an internal subset of $\mathbb{H}_1$. By the *internal counting measure* of A we shall mean the hyperreal number $\mu(A) = \#(A)/H$, where $\#(A)$ is the internal size of A. By the *Loeb measure* $L(\mu)(A)$ of A we mean the standard part $\mathrm{st}(\mu(A))$.

The set of $\mathbb{H}_1$ itself has internal size H and thus has internal counting measure 1. Recall that the set of internal subsets of $\mathbb{H}_1$ is a field of sets but is not countably additive. At this point we have only defined the Loeb measure for internal sets. This restriction of the Loeb measure is finitely additive. Our next step is to assign Loeb inner and outer measures to arbitrary subsets of $\mathbb{H}_1$.

DEFINITION. Let B be a subset of $\mathbb{H}_1$. The *inner Loeb measure* of B, $\mu_{inner}(B)$, is defined as the supremum (in the classical real sense) of the Loeb measures of all internal subsets of B. The *outer Loeb measure* of B, $\mu_{outer}(B)$, is the infimum of the Loeb measures of all internal supersets of B.

Finally, we define the Loeb measure and show that it is countably additive.

DEFINITION. A set $B \subseteq \mathbb{H}_1$ is said to be *Loeb measurable* iff $\mu_{inner}(B) = \mu_{outer}(B)$. If B is Loeb measurable, its Loeb measure is the common value of its inner and outer measure.

PROPOSITION 7. (Loeb, 1975). *The Loeb measure is a complete countably additive measure.*

Hint. The key step is to show that a countable union of sets B_n of outer Loeb measure zero has outer Loeb measure zero. Given a real $\varepsilon > 0$, for each n we may choose an internal superset $A_n \supseteq B_n$ of counting measure $\leq \varepsilon/2^n$. Now use Saturation to get a single internal set of counting measure $\leq 2\varepsilon$ which contains all of the sets A_n.

Another nice consequence of Saturation is the 'internal approximation lemma', that for every Loeb measurable set B there is an internal set A such that the symmetric difference between A and B has Loeb measure zero.

There is an elegant relationship between the Loeb measure on the hyperfinite grid and Lebesgue measure on the real line, due to Anderson (1976) and Henson (1979). For any subset C of the real interval [0, 1], C is Lebesgue measurable and only if the set $st^{-1}(C) = \{x \in \mathbb{H}_1 : st(x) \in C\}$ is Loeb measurable, and the Loeb measure of $st^{-1}(C)$ equals the Lebesgue measure of C. Here are some examples of sets which are not Loeb measurable (as usual, we omit the proofs).

Examples. (a) (Luxemburg) Let A be the internal set of all $x \in {}^*[0, 1]$ such that the Hth binary digit of x is 1. Then $A \cap [0, 1]$ is not Lebesgue measurable so $st^{-1}(A \cap [0, 1])$ is not Loeb measurable.

(b) The set $B = \{y \in \mathbb{H}_1 : y \geq st(y)\}$ is not Loeb measurable, but has the measurable standard part $st(B) = [0, 1]$.

Although the Loeb measure lives on a hyperfinite set and is amenable to finite-like computations, it is an extremely rich measure space. This richness has been exploited to prove numerous existence theorems in probability theory. For more about the subject, see the book by Albeverio *et al.* (1986).

The Loeb measurable construction can also be carried out on the whole hyperreal line instead of just the hyperfinite grid. However, the theory

is less satisfying. On the hyperfinite grid, every point has the same infinitesimal weight and every internal set is Loeb measurable. But on the hyperreal line, each point has weight zero and only some of the internal sets are Loeb measurable.

8. HYPERFINITE DESCRIPTIVE SET THEORY

Descriptive set theory on the real line deals with the Borel and projective hierarchies of subsets of $\mathbb{R}$. The open and closed sets are at the first level of the Borel hierarchy, and are called the Σ^0_1 and Π^0_1 sets, respectively. Given a countable ordinal $\alpha > 0$, the Σ^0_α sets are the countable unions of sets in $\bigcup_{\beta<\alpha}\Pi^0_\beta$, and the Π^0_α sets are the countable intersections of set in $\bigcup_{\beta<\alpha}\Sigma^0_\beta$. The collection of Borel sets is $\Sigma^1_0 = \Pi^1_0 = \bigcup_{\alpha<\omega_1}\Sigma^0_\alpha$, and the Σ^1_{n+1} and Π^1_{n+1} sets are obtained from the Π^1_n and Σ^1_n relations by existential and universal quantification.

There is a parallel descriptive set theory on a hyperfinite grid which begins with the internal sets at the initial level; see Keisler *et al.* (1989). The Saturation Principle plays a crucial role in the theory and acts as a substitute for the existence of a countable dense set on the real line. This theory has exposed both similarities and differences between the real line and the hyperfinite grid. We give a sampling of results here to build on our picture of the hyperfinite grid. Again, we simplify the discussion by restricting ourselves to the unit intervals $\mathbb{H}_1$ and [0, 1].

We take the internal subsets of the unit interval $\mathbb{H}_1$ in the hyperfinite grid to be both Σ^0_0 and Π^0_0 sets, and form the Borel hierarchy. Each monad is a Π^0_1 set, that is, a countable intersection of internal sets. The sets in the Borel hierarchy are called *Loeb sets*, and the Loeb sets form the σ-algebra generated by the internal sets. Since the Loeb measure is countably additive, every Loeb subset of $\mathbb{H}_1$ is Loeb measurable. The projective hierarchy is defined in the natural way. A starting point of the theory is the result of Robinson that a set $B \subseteq [0, 1]$ is closed if and only if $A = st(B)$ for some internal set $B \subseteq \mathbb{H}_1$.

Kunen and Miller (1983), improving an earlier result of Henson (1979), showed that the standard part inverse preserves the exact location in the hierarchies. That is, a set $B \subseteq [0, 1]$ belongs to Σ^0_α, or Σ^1_n, if and only if $st^{-1}(B)$ belongs to Σ^0_α or Σ^1_n, respectively.

Given any partial function on the hyperfinite grid whose graph is

Σ^0_α, or Π^1_n, the domain of the function is of the same class. In each case, the analogous statement fails for the real line.

The uniformization, or selection, properties for the hyperfinite grid are also different from the corresponding properties for the real line. It follows from Saturation that every Σ^0_1, or Σ^0_2, relation over the hyperfinite grid has a choice function whose graph is Σ^0_1, or Σ^0_2, respectively. The analogous statements are false over the real line. On the other hand, there is a Π^0_2 relation over the hyperfinite grid which has no Π^1_n choice function for any n. By contrast, every Π^1_1 relation over the real line has a Π^1_1 choice function.

A natural and useful notion which has no counterpart in the real line is the notion of a countably determined set (Henson, 1979). A set $B \subseteq \mathbb{H}$ is said to be *countably determined* iff B is a finite or infinite Boolean combination of some countable collection of internal sets. All the sets in the projective hierarchy Π^1_n are countably determined. A convenient way to show that a subset of $\mathbb{H}_1$ is highly complex is to show that it is not countably determined. For example, no well ordering of $\mathbb{H}_1$ is countably determined, no function from the unit interval $\mathbb{H}_1$ onto the whole grid $\mathbb{H}$ is countably determined, and no choice function for the Π^0_2 relation in the preceding paragraph is countably determined.

9. TOPOLOGY AND ORDER ON THE HYPERREAL LINE

The topology on the hyperreal line which has been used most frequently in the literature is the *S-topology*, where *S* stands for 'standard'. For simplicity, we shall restrict our attention to the hyperreal unit interval $^*[0, 1]$. A set $B \subseteq {}^*[0, 1]$ is open in the S-topology iff for each $x \in B$ there is a standard real $\varepsilon > 0$ such that each point of $^*[0, 1]$ within ε of x belongs to B. The S-topology is just the ordinary topology on $[0, 1]$ but replacing each real number by its monad. Robinson's original nonstandard characterizations of limit and continuity are formulated using the S-topology. Thus, a real function f on $[0, 1]$ is continuous if and only if $^*f$ is continuous in the S-topology.

Robinson introduced another topology on the hyperreal line, called the *Q-topology*. A set $B \subseteq {}^*[0, 1]$ is open in the Q-topology iff for each $x \in B$ there is a *hyperreal* $\varepsilon > 0$ such that each point of $^*[0, 1]$ within ε of x belongs to B.

Examples. (a) The internal function $^*\sin(H \cdot x)$ where H is positive infinite is Q-continuous but not *S*-continuous.

(b) The internal 'step' function $[x/H]$ where H is positive infinite is S-continuous but not Q-continuous.

We can get more insight into the nature of the hyperfinite line by studying its Dedekind cuts. This leads to a collection of order topologies associated with cuts. The hyperreal line is studied from this viewpoint, for example, in (Zakon, 1969; and Keisler and Leth, 1991); here we shall only mention some simple observations and examples.

DEFINITION. By a *cut* in the hyperreal unit interval $^*[0, 1]$ we shall mean a nontrivial initial segment C of $^*[0, 1]$ such that C has no greatest element and its complement has no least element. A cut is said to be *additive* iff it is closed under addition.

By the Overspill Principle, there are no internal cuts.

A cut C is said to be *regular* iff for every $x > 0$ in $^*\mathbb{R}$ there exists $y \in C$ such that $x + y \notin C$. Zakon (1969) asked whether the hyperreal interval has regular cuts. It was shown by Kamo (1981) that there exist hyperreal lines with regular cuts, assuming the continuum hypothesis. Jin and Keisler (1993) proved this fact in ZFC.

Hereafter we shall concentrate on the additive cuts. Clearly, no additive cut is regular. Additive cuts are of special interest because each additive cut induces a topology on the hyperreal line.

DEFINITION. Let C be an additive cut in $^*[0, 1]$. By the *C-monad* of a point $x \in {}^*[0, 1]$ we mean the set of all $y \in {}^*[0, 1]$ such that $|x - y| \in C$. the C-topology on $^*[0, 1]$ is defined by calling a set $B \subseteq {}^*[0, 1]$ *C-open* iff for every $x \in B$, there exists $\varepsilon \notin C$ such that each point of $^*[0, 1]$ within ε of x belongs to B.

The C-topology is not Hausdorff, because two points in the same C-monad belong to the same C-open sets. However, if we identify all the points in the same C-monad, we obtain a Hausdorff topology, which is just the order topology on the C-monads.

Examples. (a) The largest additive cut in $^*[0, 1]$ is the set of infinitesimals, and the corresponding topology is the S-topology. For any $x \in {}^*(0, 1]$, there is a greatest additive cut below x, namely the set C_x of all $z \in {}^*[0, 1]$ with $z/x \approx 0$. The C_x-topology will look like $1/x$ copies of the real interval $[0, 1)$ laid end to end (with real points replaced by C_x-monads).

(b) If C is an additive cut which is not of the form C_x, the C-topology will be totally disconnected, that is, any two distinct C-monads will be separated by clopen sets. *Hint.* Given two points x and y in different C-monads, take a $b \notin C$ which is infinitesimal compared to $|y - x|$ and form the clopen set $\{z : |x - z|/b \text{ is finite}\}$. Each of the cuts in examples (c)–(e) below cannot be of the form C_x and therefore induce totally disconnected topologies.

(c) For each infinitesimal $y \in {}^*[0, 1]$ there is a least additive cut above x, namely the set C^y of all $z \in {}^*[0, 1]$ such that z/y is finite.

(d) Each increasing sequence $\langle x_\alpha : \alpha < \lambda \rangle$ indexed by a limit ordinal λ such that each $x_\alpha / x_{\alpha+1} \approx 0$ induces an additive cut C.

(e) Lightstone and Robinson (1975) considered the C-topology where C is a cut the form $\{y : y \le x^n \text{ for all } n \in \mathbb{N}\}$ for some infinitesimal x.

(f) If we relax the requirement that a cut has no greatest element and allow $\{0\}$ as an additive cut, the corresponding topology on $^*[0, 1]$ is the Q-topology. This topology is also totally disconnected.

One way to classify cuts is by their cofinality and coinitiality. The *cofinality* of a cut C is the least cardinality of a subset C which has no upper bound in C. By the *coinitiality* of C we shall mean the least cardinality of a subset of the complement of C which has no lower bound.

The greatest additive cut below x has cofinality ω, and the least additive cut above x has coinitiality ω. It follows from Saturation that each regular cut has uncountable cofinality and coinitiality. Another consequence of Saturation is that there is no cut which has both cofinality ω and coinitiality ω.

Does there exist a hyperreal line (satisfying the Transfer and Saturation Principles) such that every cut C (or every additive cut C) has either countable cofinality or countable coinitiality? Jin (1992) proved that it is consistent with ZFC that the answer is no.

There exist hyperfinite lines with the opposite property, i.e. some additive cut C has uncountable cofinality and uncountable coinitiality. *Hint.* Take an ultrapower of an ultrapower of $\mathbb{R}$, and consider the cut formed by the second ultrapower applied to the set of finite multiples of $1/H$ in the first ultrapower.

The Baire Category Theorem is an important property of the usual topology on the real interval. The next result shows that on the hyperreal interval, all of the C-topologies satisfy the Baire Category Theorem.

A set $B \subseteq {}^*[0, 1]$ is said to be *C-nowhere dense* iff each interval of length $\notin C$ contains a subinterval of length $\notin C$ which is disjoint from B. A countable union of C-nowhere dense sets is said to be *C-meager*.

PROPOSITION 8. (Hyperreal Baire Category Theorem). *For each additive cut C in* $^*[0, 1]$, *the set* $^*[0, 1]$ *is not C-meager. The set* $^*[0, 1]$ *is also not meager in the Q-topology.*

Hint. By Saturation, the intersection of a countable chain of intervals of length $\notin$ C is nonempty.

On the real line, there is an interesting interplay between the sets of Lebesgue measure zero and the meager sets. On the hyperreal line there is a similar interplay between sets of Loeb measure zero and C-meager sets. The following examples give some idea of what can happen.

Examples. (a) The unit hyperfinite grid $\mathbb{H}_1$ is C-meager if and only if $1/H \notin C$. In the remaining examples, we consider subsets of $\mathbb{H}_1$ and let C range over the additive cuts such that $1/H \in C$.

(b) The 'Cantor ternary' set, consisting of all $x \in \mathbb{H}_1$ such that $x \cdot H$ has no 1s in base 3, is C-meager for all C and has Loeb measure zero.

(c) Let P be the set of all $x \in \mathbb{H}_1$ such that $x \cdot H$ is *prime. P has Loeb measure zero. P is C-meager where C is the smallest additive cut greater than $1/H$, but is not S-meager; in facts, the family of cuts C for which P is C-meager is rather complicated.

(d) If C has either confinality ω or coinitiality ω, then there is a C-meager subset of $\mathbb{H}_1$ of Loeb measure one. *Hint.* Find a Cantor set in a hyperfinite base which is C-nowhere dense and of Loeb measure close to one. (The situation for other cuts C is more complex; see Keisler and Leth, 1991.)

(e) There is an internal subset $A \subseteq \mathbb{H}_1$ such that A has Loeb measure zero but for all C, A is not C-meager. *Hint.* Start with a real set $B \subseteq \mathbb{N}$ such that for each n, $B \cap [3^n, 3^{n+1})$ contains one interval of length 2^n and all multiples of 2^n.

10. IS THE REAL LINE UNIQUE?

A classical theorem is Zermelo set theory states that there is a unique complete ordered field up to isomorphism. Thus within any model of Zermelo set theory there is a unique real line. To a Platonist who regards the axioms of set theory as describing a universe of sets which really exists, the real line is unique.

In fact, the unique existence of the real line can be proved in any reasonable set theory which has the power set axiom. We may thus think of the real line as existing uniquely *relative to the power set operation.*

However, there is a case against the view that the real line is unique. Unlike the natural numbers, the real line is not absolute for transitive models $\langle M, \in \rangle$, and different transitive models of set theory have markedly different real lines. By Tarski's theorem, the first-order theory of the real line is the same in all models of Zermelo set theory. However, the second- and higher-order theories of the real line depend on the underlying universe of set theory. The method of forcing has produced innumerable examples of statements in the second-order logic of the real line which are independent of Zermelo set theory and various stronger set theories. Thus the *properties* of the real line are not uniquely determined by the axioms of set theory. To a mathematician who doubts the existence of the set theoretic universe, or who believes that several competing set theoretic universes exist, the properties of the real line need not be unique.

Historically, Zermelo set theory was developed in order to form a rigorous foundation for the real line. A set theory which was not strong enough to prove the unique existence of the real line would not have gained acceptance as a mathematical foundation. The modern set theory KPU, Kripke-Platek set theory with urelements, is an example of a set theory in which the real line cannot be proved to exist. KPU is a natural foundation for recursion theory over sets; a good reference is the book by Barwise (1975).

11. IS THE HYPERREAL LINE UNIQUE?

The existence and uniqueness of the hyperreal line depends on the choice of the underlying set theory and on how one defines the notion of a hyperreal line.

We shall first discuss the situation in ordinary Zermelo or Zermelo-Fraenkel set theory, with the hyperreal line defined to be an ultrapower of the real line. In ZFC, the existence of the hyperreal number system requires the axiom of choice in two places, first to obtain an ultrafilter which is not closed under countable intersections and then to prove the Transfer Principle (but see Luxemburg, 1962; Pincus, 1974; and Spector, 1988, for applications and a theory of ultrapowers without the axiom of choice). Given the axiom of choice, there are many different ultrafilters which give rise to difference hyperreal lines. Thus if we take our underlying set theory to be Zermelo or Zermelo-Fraenkel set theory with choice, the hyperreal line can be proved to exist but not to be unique.

One can obtain uniqueness up to isomorphism by strengthening ZFC and the definition of a hyperreal number system. By a *fully saturated* hyperreal number system we shall mean an elementary extension $\langle {}^*\mathbb{R}, {}^*S : S \in \mathcal{F} \rangle$ of the full structure over $\mathbb{R}$ such that any set of fewer than card $({}^*\mathbb{R})$ formulas with constants from ${}^*\mathbb{R}$ which is finitely satisfiable is satisfiable. It follows from results of the Morley and Vaught (1962) that in ZFC, there exists a unique fully saturated hyperreal number system in any cardinal $\kappa > 2^\omega$ such that either κ is inaccessible or $\kappa = \lambda^+ = 2^\lambda$ (i.e. the GCH holds at κ). In the second case $\kappa = \lambda^+ = 2^\lambda$, it follows from (Keisler, 1965) that the fully saturated hyperreal number system of cardinality κ is an ultrapower of $\mathbb{R}$. In the case that κ is inaccessible, it is open whether a fully saturated hyperreal number system, or even any model of cardinality κ, can be an ultrapower of $\mathbb{R}$.

However, it may not be a good idea to restrict the notion of a hyperreal number system by requiring full saturation. In some applications of the hyperreal number system, full saturation of large cardinality has proved to be a desirable hypothesis. But there are other applications where different hypotheses on the hyperreal number system are needed (for example, it is sometimes useful to take the hyperreal number system to be an ultrapower of $\mathbb{R}$ modulo a selective ultrafilter over $\mathbb{N}$). It is better to leave open the possibility of adding a variety of extra hypotheses on the hyperreal number system.

In the case of the real number system, the real line is unique relative to the underlying set theory. The second order theory of the real line is not unique, and this 'absolute' non-uniqueness is sometimes useful. However, the non-uniqueness is exploited by adding extra axioms to the underlying set theory, while keeping the definition of the real number system within the set theory fixed.

This suggests that ZFC is not the appropriate underlying set theory for the hyperreal number system. Set theory might have taken a different direction if it had been developed with the hyperreal line in mind. What is needed is an underlying set theory which proves the unique existence of the hyperreal number system, with the possibility of exploiting the absolute non-uniqueness by adding extra axioms in the same manner as is done for the real number system in ZFC. This underlying set theory should have the power set operation to insure the unique existence of the real number system, and another operation which insures the unique existence of the pair consisting of the real and hyperreal number systems.

12. HYPERREAL SET THEORIES

We have concentrated on the hyperreal line in this article, but we shall now shift to the broader perspective of a 'nonstandard universe' where the hyperreal line appears at the first level. Several set theories have been proposed as foundations for a nonstandard universe. All such theories have a Transfer Principle which guarantees that the hyperreal line is an elementary extension of the full structure on the real line. In this broader perspective, the hyperreal line is not necessarily an ultrapower of the real line, but by the results of (Keisler, 1963) it must be a limit ultrapower. Two approaches, which we shall discuss briefly here, are currently used in the literature.

The first approach is now called the *superstructure approach.* It is related to Robinson's original formulation in (Robinson, 1966) using the theory of types, and is due to Robinson and Zakon (1969). It is often presented by constructing a model within ZFC, but we shall formulate it axiomatically. We shall give this theory the name RZ.

We first motivate the theory by describing its intended interpretation. Given a set X, we inductively define

$$V_0(X) = X, \quad V_{n+1}(X) = X \cup V_n(X), \quad V(X) = \cup_{n\in\mathbb{N}} V_n(X).$$

$V(X)$ is called the *superstructure* over X. The intended models of RZ are structures of the form

$$\langle V(X), V(Y), *\rangle$$

where X and Y are nonempty sets and $* : V(X) \to V(Y)$ is an embedding which preserves first-order formulas in the vocabulary $\{=, \in\}$ with only bounded quantifiers $(\forall u \in v)$, $(\exists u \in v)$. (To avoid unintended $\in$ relationships, X must be chosen so that $\varnothing \notin X$ and each $x \in X$ is disjoint from $V(X)$, and similarly for Y). The sets in $V(X)$ are called *standard* sets, the sets in $V(Y)$ are called *external* sets, and the sets in $V(Y)$ which are elements of $^*A$ for some $A \in V(X)$ are called *internal* sets.

The language of RZ is a two-sorted predicate logic with equality, unary predicate symbols X in the first (standard) sort and Y in the second (external) sort, a binary relation symbol $\in$ for each sort, and a function symbol $*$ whose type is a mapping from the first sort into the second sort. Let Z denote Zermelo set theory with choice and urelements (individuals) but without the axiom of infinity. The axioms of RZ will consist of one copy of the theory Z for each sort with X and Y the sets of urelements, a scheme which says that $*$ is an elementary embedding with respect to bounded quantifier formulas, and an axiom stating that X has a countable subset N and $*$ maps N properly into $^*N$.

Both the real and hyperreal number systems exist uniquely relative to the theory RZ. In RZ, one can prove that X contains a copy of $\mathbb{N}$ (unique up to isomorphism). Using $\mathbb{N}$ and the power set operation, one can then prove in the usual way that there is a unique (up to isomorphism) complete ordered field $\mathbb{R}$ at level 1 of the first sort. The hyperreal number system is then characterized uniquely up to isomorphism as the structure $\langle {}^*\mathbb{R}, {}^*S : S \in \mathcal{F}\rangle$ in the second sort.

In RZ, the property 'B is internal' is definable by the formula $(\exists A)\, B \in {}^*A$. The Saturation Principle is expressible in RZ as follows: For every internal set C, every countable chain B of internal subsets of C has a nonempty intersection. Other properties, such as saturation for higher cardinals, are also expressible in RZ and thus can be added as extra axioms if necessary.

Using the ultrapower construction, ZFC proves that RZ is consistent and in fact every superstructure $\langle V(X), \in\rangle$ with an infinite X can be expanded to a transitive model of RZ. Therefore any statement about the standard superstructure which can be proved in RZ can also be proved in ZFC and hence is a theorem according to the usual rules of mathematics.

The theory RZ would become inconsistent if the replacement scheme were added (upgrading the Zermelo axioms to the Zermelo-Fraenkel axioms). *Hint.* Show that every standard set $^*A$ has finite rank, because otherwise there would be an infinite decreasing $\in$ – sequence below $^*A$.

On the other hand, all of classical mathematics can be carried out in each of the two superstructures in RZ. Moreover, the interaction between the standard, internal, and external sets and the availability of the Zermelo axioms for the external as well as the standard sets makes RZ quite powerful.

A second approach is Nelson's Internal Set Theory (IST), introduced in (Nelson, 1977). IST has the equality and $\in$ symbols and one additional unary predicate symbol st(·) for *standardness*. The axioms of IST are the axioms of ZFC, the Internalization scheme which is a weak form of Saturation, the Standardization scheme which says that the restriction of any formula to a standard set defines a standard set, and the Transfer scheme which says that the standard sets form an elementary submodel of the universe with respect to the vocabulary $\{=, \in\}$ of ZFC.

Compared to RZ, IST retains the replacement scheme of ZFC, but gives up the external sets, leaving only the internal and standard sets. Since IST is an extension of ZFC and uses the st(·) predicate instead of the $*$ function, mathematics in IST looks more like traditional mathematics than mathematics in RZ does. For this reason, it is easier for a classical mathematician to read work in IST than in RZ. However, because the external sets are missing, developments such as the Loeb measure construction and hyperfinite descriptive set theory cannot be carried out in their full generality in IST.

In IST, the usual construction of the real line $\mathbb{R}$ gives us the hyperreal line instead, and the restriction of the hyperreal line to the standard predicate gives us the real line. Thus the real and hyperreal lines exist uniquely relative to IST.

Using a limit ultrapower construction, Nelson (1977) proved that IST is a conservative extension of ZFC. Therefore, as in the case of RZ, any sentence in the language of ZFC (without the st(·) predicate) which is provable in IST is already provable in ZFC and hence is a theorem according to the usual rules of mathematics.

The conservative extension results are often misinterpreted as saying that anything that can be proved with Robinson's analysis can be proved without it. This issue is taken up in (Henson *et al.*, 1984; Henson and Keisler, 1986). As explained, for example, in (Simpson, 1984), almost all of classical mathematics uses only a small part of ZFC, at most second order arithmetic with Π^1_1 comprehension. In (Henson and Keisler, 1986) it is shown that Robinson's analysis uses principles beyond Π^1_1 comprehension in a natural way, thus bringing more of ZFC within the reach of our intuition.

13. DO WE LIVE IN A HYPERFINITE UNIVERSE?

Our intuitive concept of a geometric line is based upon a finite amount of experience with a line in physical space. From this finite experience, we have no way to determine its microscopic structure. For example, we cannot tell whether it is finite or infinite, whether it has the Archimedean property, or whether or not it is discretely ordered. Without direct physical evidence, we must fall back on less direct criteria for choosing a mathematical model for the geometric line.

One can take a Platonistic view that the geometric line exists, look for properties which the geometric line should have, and represent the geometric line by a mathematical object which has these properties. Alternatively, one can take a pragmatic approach and look for mathematical lines which are useful in explaining or modeling natural phenomena, or in the discovery of mathematical results.

One Platonistic view is that the geometric line should be as rich as possible, in some sense containing all possible points. With this criterion, the geometric line should be a fully saturated hyperreal line of large cardinality. Taking this to the extreme, the geometric line should be a hyperreal line which is a fully saturated model whose universe is a proper class. A similar but more convenient alternative is to take the geometric line to be a hyperreal line which is fully saturated model whose size is an uncountable inaccessible cardinal.

Another Platonistic view is that the geometric line should be rich but should also be as much as possible like the large finite lines which we know from experience. This leads to the hyperfinite grid. On the hyperfinite grid, Zeno's Paradox is resolved as follows. We can get from 0 to 1 in H steps by taking one step of length $1/H$ every $1/H$ seconds, always staying in the hyperfinite grid $\mathbb{H}$. Along the way, we will pass through all the points 1/2, 3/4, 7/8, and so on, since they all belong to the set $\mathbb{H}$. Of course, we will overshoot irrational points such as $\sqrt{2}/2$, but there will be a time at which we pass from below $\sqrt{2}/2$ to above $\sqrt{2}/2$ with one step of length $1/H$.

A pragmatic argument for the hyperreal line is that it provides a useful source of models for many natural phenomena. There is an extensive literature in mathematical economics (see Rashid, 1987) and physics (see Albeverio *et al.*, 1986) using the hyperreal line. In microeconomics, one studies the behavior of economies with a large number of individually small agents. The large finite economy is represented by a mathematically simpler infinite economy. Large finite economies are sometimes modeled by economies with a continuum of agents, but since the original economy is finite, a hyperfinite set of infinitesimal agents provides a better model than a continuum of agents. A similar approach is useful in physics, where, for example, a large finite set of small particles can be modeled by a hyperfinite set of infinitesimal particles. In some cases, as in the work of Arkeryd (1984) on the Boltzmann equation and the work of Cutland (1986) in control theory, the real line is not rich enough to provide a mathematical representation of a physical phenomenon, and the richer hyperreal line comes to the rescue. The hyperreal line is also helpful in representing phenomena with two scales of measurement where one scale is very large compared to the other. For example, Reeb and his students (see F. and M. Diener (1988) or van den Berg (1987)) have studied singular perturbations by looking through an infinitesimal microscope to classify hyperreal solutions whose trajectories have infinitely fast and slow parts.

In physics, the evidence for the existence of an object such as a quark is indirect, and often the only evidence is that the object makes it easier to mathematically represent an observed phenomenon. The hyperreal line makes it easier to mathematically represent natural phenomena, and this may be taken as evidence that the hyperreal line exists in some sense.

A second pragmatic argument for the hyperreal line is that it is helpful in the process of mathematical discovery. There are many examples where it has either suggested a fruitful new notion, been used to prove a new result, or been used to give a clearer proof of an old result.

The following strategy, sometimes called the *lifting method*, has been used to prove results which are formulated on the ordinary real line.

Step 1. Lift the given 'real' objects up to internal approximations on the hyperfinite grid.

Step 2. Make a series of hyperfinite computations to construct some new internal object on the hyperfinite grid.

Step 3. Come back down to the real line by taking standard parts of the results of the computations.

The hyperfinite computations in Step 2 will typically replace more problematic infinite computations on the real line. Usually, the hard work is in Step 3, where one must show that the appropriate standard parts exist. As a typical example, various existence theorems for stochastic differential equations have been solved by lifting up to the hyperreal line, easily solving the corresponding stochastic difference equation, and then taking standard parts to obtain a solution of the original stochastic differential equation.

To date, the lifting strategy has been fully exploited in two areas, probability theory and Banach spaces (see (Albeverio *et al.*, 1986; Cutland, 1988; Keisler, 1984) for a variety of applications of this strategy). However, there appear to be possibilities for its use in practically all areas of mathematics.

We have stated that the hyperreal line can be used to give a clearer proof of a result. One reason for this is that hyperreal proofs seem to be more 'constructive' than classical proofs. For example, the solution of a stochastic differential equation given by the hyperreal proof is obtained by solving a hyperfinite difference equation by a simple induction and taking the standard part. The hyperreal proof is not constructive in the usual sense, because in ZFC the axiom of choice is needed even to get the existence of a hyperreal line. What often happens is that a proof within RZ or IST of a statement of the form $\exists x\phi(x)$ will produce an x which is *definable from H*, where H is an arbitrary infinite hypernatural number. Thus instead of a pure existence proof, one obtains an explicit solution except for the dependence on H. The extra information one gets from the explicit construction of the solution from H makes the proof easier to understand and may lead to additional results.

Where do we go from here? At the present time, the hyperreal number system is regarded as somewhat of a novelty. But because of its broad potential, it may eventually become a part of the basic toolkit of mathematicians. This process will probably take a very long time, perhaps 50 to 100 years. The current high degree of specialization in mathematics serves to inhibit the process, since few established mathematicians are willing to take the time to learn both mathematical logic and an area of application. However, in the long term, applications of mathematical logic to computer science, as well as applications of the hyperreal numbers, should result in future generations of mathematicians who are better trained in logic, and therefore more able to take advantage of the hyperreal line when the opportunity arises.

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