Zeno's Metrical Paradox of Extension

ADOLF GRÜNBAUM

§1. THE PROBLEM

It is a commonplace in the analytic geometry of physical space and time that an *extended* straight-line segment, having positive length, is treated as "consisting of" *unextended* points, each of which has zero length. Analogously, time intervals of positive duration are postulated to be aggregates of instants, each of which has zero duration.

Ever since some of the Greeks defined a point as "that which has no part,"[1] philosophers and mathematicians have questioned the consistency of conceiving of an extended continuum as an aggregate of unextended elements. On the long list of investigators who have examined this question in the context of the specific mathematical and philosophical theories of their time, we find not only Zeno[2] but also such

From *Modern Science and Zeno's Paradoxes* (Middletown, Conn.: Wesleyan University Press, 1967), pp. 115–135. Reprinted by permission. British edition, revised (London: George Allen & Unwin Ltd., 1968).

[1] This definition is given in Euclid, *The Thirteen Books of Euclid's Elements*, translated by T. L. Heath (New York: Cambridge University Press, 1926), p. 153.

[2] S. Luria, "Die Infinitesimaltheorie der antiken Atomisten," *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik*, Abteilung B, Studien II (Berlin, 1933), p. 106.

thinkers as Aristotle,[3] Cavalieri,[4] Tacquet,[5] Pascal,[6] Bolzano,[7] Leibniz,[8] Paul du Bois-Reymond,[9] and Georg Cantor,[10] to mention but a few. Thus, William James wrote:

> If, however, we take time and space as concepts, not as perceptual data, we don't well see how they can have this atomistic constitution. For if the drops or atoms are themselves without duration or extension it is inconceivable that by adding any number of them together times or spaces should accrue.[11]
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> . . . that being should be identified with the consummation of an endless chain of units (such as "points"), no one of which contains any amount whatever of the being (such as "space") expected to result, this is something which our intellect not only fails to understand, but which it finds absurd.[12]

Writing on this issue more recently, P. W. Bridgman declared:

> With regard to the paradoxes of Zeno . . . if I literally thought of a line as consisting of an assemblage of points of zero length and of an interval of time as the sum of moments without duration, paradox would then present itself.[13]

[3] Aristotle, *On Generation and Corruption*, Book I, Chapter ii, 316a15–317a17; A. Edel, *Aristotle's Theory of the Infinite* (New York: Columbia University Press, 1934), pp. 48–49, 76–78; T. L. Heath, *Mathematics in Aristotle* [14], pp. 90, 117.

[4] C. B. Boyer, *The Concepts of the Calculus* (New York: Hafner Publishing Co., Inc. 1949), p. 140. [Dover edn. [31]].

[5] *Ibid.*

[6] *Ibid.*, p. 152.

[7] *Ibid.*, p. 270; and B. Bolzano, *Paradoxes of the Infinite*, edited by D. A. Steele (New Haven: Yale University Press, 1951).

[8] B. Russell, *The Philosophy of Leibniz* (London: George Allen & Unwin Ltd., 1937), p. 114.

[9] P. du Bois-Reymond, *Die Allgemeine Funktionentheorie*, Vol. I (Tübingen: Lauppische Buchhandlung, 1882), p. 66.

[10] G. Cantor, *Gesammelte Abhandlungen*, edited by E. Zermelo (Berlin: Springer-Verlag, 1932), pp. 275, 374.

[11] W. James, *Some Problems of Philosophy* [44], p. 155.

[12] *Ibid.*, p. 186.

[13] P. W. Bridgman, "Some Implications of Recent Points of View in Physics," *Revue Internationale de Philosophie*, III, No. 10 (1949), p. 490.

This Zenonian criticism of the mathematical theory of physical space and time by James and Bridgman is a challenge to the basic Cantorean conceptions underlying analytic geometry and the mathematical theory of motion.[14] Their view also calls into question such philosophies of science as rely on these conceptions for the interpretation of our mathematical knowledge of nature. Accordingly, it is essential that we inquire whether contemporary point-set theory succeeds in avoiding an inconsistency upon postulating positive linear intervals to be aggregates of extensionless point-elements.

In the present chapter I shall endeavor to exhibit those features of present mathematical theory which do indeed preclude the existence of such an inconsistency. It will then be clear what kind of mathematical and philosophical theory does succeed in avoiding Zeno's mathematical (metrical) paradoxes of plurality, paradoxes that I have distinguished from his paradoxes of motion in the Introduction.* As before, my concern with the views which various writers have attributed to Zeno is exclusively systematic, and I make no claims whatever regarding the historicity of Zeno's arguments or concerning the authenticity of views which I shall associate with his name. According to S. Luria,[15] Zeno invokes two basic axioms in his mathematical paradoxes of plurality. Having divided all magnitudes into positive and "dimensionless," i.e., unextended magnitudes, Zeno assumed that (1) the sum of an infinite number of equal positive magnitudes of arbitrary smallness must necessarily be infinite, and (2) the sum of *any* finite or *infinite* number of "dimensionless" magnitudes must necessarily be *zero*.

The second of these axioms seems to command the assent of P. W. Bridgman and was also enunciated by the mathematician Paul du Bois-Reymond,[16] who then inferred that we

[14] G. Cantor, *Gesammelte Abhandlungen*, p. 275.
*[In *Modern Science and Zeno's Paradoxes*, p. 3.]
[15] S. Luria, "Die Infinitesimaltheorie der antiken Atomisten," p. 66.
[16] P. du Bois-Reymond, *Die Allgemeine Funktionentheorie*, Vol. I, p. 66.

cannot regard a line as an aggregate of "dimensionless" points, however dense an order we postulate for this aggregate. Zeno himself is presumed to have used these axioms as a basis for the following dilemma:[17] If a line segment is postulated to be an aggregate of infinitely many *like* elements, then two and only two cases are possible. Either these elements are of equal positive length and the aggregate of them is of infinite length (by Axiom 1) or the elements are of zero length and then their aggregate is necessarily of zero length (by Axiom 2). The first horn of this dilemma is valid but does not have relevance to the modern analytic geometry of space and time. It is the second horn that we must refute in the context of present mathematical theory if we are to solve the problem which we have posed.

To carry out this refutation, we must first ascertain the logical relationships between the modern concepts of metric, length, measure, and cardinality, when applied to (infinite) point-sets. For in the second horn of his dilemma, Zeno avers that a line cannot be regarded as an aggregate of points no matter what cardinality we postulate for the aggregate. And du Bois-Reymond endorsed this contention by reminding us that points are "dimensionless," i.e., unextended, and by maintaining that if we conceive the line to be "merely an aggregate of points" then we are eo *ipso* abandoning the view that "A line and a point are entirely different things."[18]

We see that du Bois-Reymond is conforming to the long intuitive tradition of using the concepts of length and dimensionality interchangeably to characterize (sensed) extension. It will therefore be best to begin our analysis by noting that we must distinguish the traditional *metrical* usage of the term "dimensionless" from the contemporary *topological* meaning of "zero dimension." This distinction has become necessary by virtue of the autonomous development of the topological theory of dimension apart from metrical geome-

[17] H. Hasse and H. Scholz, *Die Grundlagenkrisis der griechischen Mathematik* (Charlottenburg: Pan-Verlag, 1928), p. 11.
[18] P. du Bois-Reymond, *Die Allgemeine Funktionentheorie*, Vol. I, p. 65.

try. Prior to this development, any positive interval of Cartesian *n*-space was simply called "*n*-dimensional" by definition. Thus, line segments having length were called "one-dimensional" and surfaces having area "two-dimensional." By contrast, in the topological theory of dimension developed in the present century, it is a non-trivial *theorem* that lines are topologically one-dimensional, surfaces two-dimensional, and, generally, that Cartesian *n*-space is *n*-dimensional. In fact, it is this theorem which warrants the use of the name "dimension theory" for the branch of topology dealing with such non-metrical properties of point-sets as make for the validity of this theorem.[19]

By contrast, the traditional *metrical* sense of dimensionality identifies dimensionality with length or measure of extendedness. It is only the latter sense of "dimension" and "dimensionless" which is relevant to the metrical problem of this chapter. Hence I refer the reader to another publication[20] for an account of how the twentieth-century theory of dimension can *consistently* affirm the following *additivity* properties for dimension in the *topological* sense of "zero-dimensional" and "one-dimensional": The point-set constituting the number axis or any finite interval in it (e.g., an infinite straight line or a finite line segment, respectively) is *one*-dimensional even though it is the set-theoretic sum of *zero*-dimensional subsets. The zero-dimensional subsets are: (1) any unit point-set (such a set has a single point as its only member and hence can be loosely referred to as a "point," whenever such usage does not permit ambiguities), (2) any *finite* collection of one or more points, (3) any denumerable set (in particular the set of rational real points), and (4) the set of irrational real points, which is non-denumerably infinite.

Accordingly, we must now deal with the following *metrical* question: Within the framework of the standard mathematics used in physics, how can the definition of length

[19] K. Menger, *Dimensionstheorie* (Leipzig: B. G. Teubner, 1928), p. 244.

[20] A. Grünbaum, "A Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements" [69], pp. 290–295.

consistently assign zero length to *unit* point-sets or individual points while assigning positive finite lengths to such unions (sums) of these unit point-sets as constitute a finite interval? To furnish an answer to the latter question will be to refute the second horn of Zeno's dilemma. We shall furnish an analysis satisfying these requirements after devoting some attention to the consideration of prior related problems.

§2. THE ADDITIVITY OF LENGTH AND MEASURE

Length, measure, or extension is defined as a property of point-sets rather than of individual points, and zero length is assigned to the *unit set*, i.e., to a set containing only a single point. While it is both logically correct and even of central importance to our problem that we treat a line interval of geometry as a set of point-elements, strictly speaking the definition of "length" renders it incorrect to refer to such an interval as an "aggregate of unextended points." For the properties of being extended or being unextended each characterize unit point-*sets* but are not possessed by their respective individual point-elements, much as temperature is a property only of aggregates of molecules and not of individual molecules. The entities which can therefore be properly said to be unextended are *included in* but are not *members of* the aggregate of points constituting a line interval. Accordingly, the line interval is a union of unextended unit point-sets and, strictly, not an "aggregate of unextended points." Though strictly incorrect, I wish to use the latter designation in order to avoid the more cumbersome expression "union of unextended unit point-sets."

I shall now present such portions of the theory of metric spaces as bear immediately on our problem.

The structure characterizing the class of all real numbers (positive, negative, and zero) arranged in order of magnitude is that of a linear Cantorean continuum.[21]

[21] E. V. Huntington, *The Continuum and Other Types of Serial Order*, 2nd edition (Cambridge: Harvard University Press, 1942), pp. 10, 44. [Dover edn. [126].]

The Euclidean point-sets or "spaces" which we shall have occasion to consider are "metric" in the following complex sense:[22]

1) There is a one-to-one correspondence between the points of an n-dimensional Euclidean space E^n and a certain real coordinate system $(x_1, \ldots, x_n)$.

2) If the points x, y have the coordinates x_i, y_i, then there is a real function $d(x, y)$, called their (Euclidean) *distance*, given by

$$d(x, y) = \left[\sum_{1}^{n} (x_i - y_i)^2 \right]^{\frac{1}{2}}$$

The basic properties of this function are given by certain distance *axioms*.[23]

A finite interval on a straight line is the (ordered) set of all real points between (and sometimes including one or both of) two fixed points called the "end-points" of the interval. Since the points constituting an interval satisfy condition (1) above in the definition of "metric," it is possible to define the "distance" between the fixed end-points of a given interval. The *number* representing this distance is the *length* of the point-set constituting the interval. Let "a" and "b" denote, respectively, the points a and b *or* their respective real-number coordinates, depending upon the context. We then define the length of a finite interval (a, b) as the non-negative quantity $b - a$, regardless of whether the interval $\{x\}$ is closed ($a \leqq x \leqq b$), open ($a < x < b$), or half-open ($a \leqq x < b$ or $a < x \leqq b$). (It is understood that the symbols "<" and "=" have a purely ordinal meaning here.) Therefore, the set-theoretic addition of a single point to an open interval (or to a half-open interval at the open end) has no effect at all on the *length* of the resulting interval as compared with the length of the original interval. In the limiting case of $a = b$,

[22] S. Lefschetz, *Introduction to Topology* (Princeton, N.J.: Princeton University Press, 1949), p. 28.

[23] *Ibid.*

the interval is called "degenerate," and here the closed interval reduces to a set containing the single point $x = a$, while each of the other three intervals is empty. *It follows that the length of a degenerate interval is zero.* Loosely speaking, a single point has zero length.[24]

Zeno is challenging us to obtain a result differing from zero when using the additivity of lengths to determine the length of a finite interval on the basis of the known zero lengths of its degenerate subintervals, each of which has a single point as its only member. But since each positive interval has a *non-denumerable* infinity of degenerate subintervals, we see already that the result of determining the length of that interval by "compounding," in some unspecified way, the zero lengths of its degenerate subintervals is far less obvious than it must have seemed to Zeno, who did not distinguish between countably and non-countably infinite sets!

Although length is similar to cardinality in being a property of *sets* and not of the elements of these, it is essential to realize that the cardinality of an interval is not a function of the length of that interval. The independence of cardinality and length becomes demonstrable by combining our definition of length with Cantor's proof of the equivalence of the set of all real points between 0 and 1 with the set of all real points between *any* two fixed points on the number axis. It is therefore not the case that the longer of two positive intervals has "more" points. In the case of two unequal intervals, one of which is a proper part of the other, the longer interval contains points which are *not* also contained in the shorter one. In this latter sense of the specified *difference in the identity and comprehensiveness of membership*, the longer of two such intervals may be said to contain "more" points, i.e., points other than the points belonging to the shorter interval. But this "more" of differing identity and comprehensiveness must *not* be confused with the

[24] H. Cramér, *Mathematical Methods of Statistics* [134], pp. 11, 19.

"more" of greater numerosity (cardinality). And it is the specified kind of greater comprehensiveness that makes for greater spatial (or temporal) extension.

Once the independence of cardinality and length of intervals is established, it is possible to eliminate several of the confusions which have vitiated certain treatments of the infinite divisibility of intervals, as we shall see below. Thus, it will become impossible to infer in finitist manner that the division of an interval into two or more subintervals imparts to each of the resulting subintervals a cardinality lower than the cardinality of the original interval.

An interesting illustration of the independence of cardinality and length is provided by the so-called "ternary set" (Cantor discontinuum). This set has measure zero (and zero dimension) while having the cardinality of the continuum.[25] And the existence of this set shows that the cardinality as such is *not* sufficient to confer positive extension on *an interval* but that its positive extension depends on the *structural arrangement* of its elements.

We shall be concerned with ascertaining why Zeno's *paradoxical* result that the length of a given *positive* interval (a, b) is *zero* is *not* deducible from the following two propositions in our geometry in the context of its rules governing the additivity of lengths: (1) Any positive or non-degenerate interval is the union of a continuum of degenerate subintervals, and (2) the length of a degenerate (sub) interval is zero. It is obvious that *if the theory is consistent,* Zeno's result cannot be deducible. Such a result would contradict the proposition that the length of the interval (a, b) is $b - a$ $(a \neq b)$. Furthermore, this result would be incompatible with Cantor's theorem that all positive intervals have the same cardinality regardless of length, for this theorem shows that no inference regarding the length of a non-degenerate interval can be drawn from propositions (1) and (2) via the ad-

[25] R. Courant and H. Robbins, *What Is Mathematics?* [133], p. 249. Also, A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev, *Mathematics*, Vol. III, translated by K. Hirsch (Cambridge: The M.I.T. Press, 1963), pp. 24 and 28.

ditivity of lengths permitted by the theory. In order to show later that the standard mathematical theory used in physics does have the required consistency, i.e., that it does not lend itself to the deduction of Zeno's paradoxical result, we must now consider the determination of (1) the length of the union of a *finite* number of non-overlapping intervals of known lengths, and (2) the length of the union of a *denumerable* infinity of such intervals.

If an interval i is the union of a *finite* number of intervals, no two of which have a common point, i.e., if

$$i = i_1 + i_2 + i_3 + \ldots + i_n \qquad (i_p i_q = 0 \text{ for } p \neq q),$$

it follows readily from the theory previously developed that the length $b - a$ of the total interval is equal to the *arithmetic sum* of the individual lengths of the subintervals. We therefore write

$$L(i) = L(i_1) + L(i_2) + L(i_3) + \ldots + L(i_n).$$

If we now *define* the arithmetic sum of a progression of finite cardinal numbers as the limit of a sequence of partial arithmetic sums of members of the sequence, then a nontrivial proof can be given[26] that the following theorem holds: The length of an interval which is subdivided into an *enumerable* number of subintervals without common points is equal to the arithmetic sum of the lengths of these subintervals.[27] It follows at once that if the standard mathematical theory containing this result *were* to assert as well—which it does *not!*—that an interval consists of an enumerable number of points, then Zeno's paradox would be deducible.

Thus, both for a finite number and for a countably infinite number of non-overlapping subintervals, the length $L(i)$ of the total interval is an additive function of the interval i. The length of an interval is a numerical measure of the comprehensiveness (extension) of that interval's *membership* relative to the standard of length but not of its cardinality. The latter

[26] Cf. H. Cramér, *Mathematical Methods of Statistics* [134], pp. 19–21.

[27] See also the discussion in [*Modern Science and Zeno's Paradoxes*] Chapter II, §2A.

does not depend upon the comprehensiveness of the membership of an interval.

It will be recalled that "length" was defined only for intervals. So far, we have not assigned any property akin to length to other kinds of point-sets. There are many occasions, however, when it is desirable to have some kind of *measure* of the extensiveness, as it were, of point-sets quite different from intervals. Problems of this kind as well as problems encountered in the theory of (Lebesgue) integration have prompted the introduction of the generalized metrical concept of "measure" $L(S)$ of a set S to deal as well with sets other than intervals. This metrical concept extends the definition of the interval function $L(i)$ so as to obtain a non-negative and additive set function $L(S)$ which coincides with $L(i)$ in the special case when S is an interval i. And the principles of the resulting measure theory relevant to our concern with Zeno's metrical paradox are the following:

> 1) The measure of a set of points is to be a number dependent on the set, such that the measure of the sum of two sets, which have no point in common, is the sum of the measures of the two sets. . . . The measure of a set being regarded as a function of the set, is thus required to be an *additive* function, i.e., a function such that its value for the set $E_1 + E_2$ is the sum of its values for E_1 and E_2.[28]
>
> 2) . . . any sum . . . of a finite or enumerable number of measurable sets [all contained in a non-infinite interval] is itself measurable.[29]
>
> 3) The measure of the sum of an enumerably infinite sequence of sets, no two of which have a point in common, is to be the limiting sum of the measures of the sets, whenever that limiting sum exists.[30]
>
> 4) Every enumerable set of points is measurable, and its measure is zero.[31]

[28] E. W. Hobson, *The Theory of Functions of a Real Variable*, Vol. I [New York: Dover Publications, Inc., 1953], p. 166.
[29] H. Cramér, *Mathematical Methods of Statistics* [134], p. 32.
[30] E. W. Hobson, *The Theory of Functions of a Real Variable*, Vol. I, p. 166.
[31] *Ibid.*, p. 176.

It will be noted that in virtue of (2) and (3), the standard mathematical theory asserts that the measure is *countably additive* (or enumerably additive), just as it had asserted for length, as is evident from our earlier discussion of the additivity of length.[32]

Since the theory of infinite divisibility has been used fallaciously in an attempt to deduce Zeno's metrical paradox, we shall now point out the relevant fallacies before dealing with the crux of our problem to refute the second horn of Zeno's metrical dilemma.

§3. INFINITE DIVISIBILITY

In an exchange of views with Leibniz, Johann Bernoulli committed an important fallacy: He treated the actually infinite set of natural numbers as having a *last* or "∞th" term which can be "reached" in the manner in which an inductive cardinal can be reached by starting from zero.[33] Bernoulli's view is clearly self-contradictory, since no such discrete denumerable infinity of terms could possibly have a last term.

When giving arguments in behalf of his theory of infinitesimals, C. S. Peirce[34] committed the same Bernoullian fallacy by reasoning as follows: (1) The decimal expansion of an irrational number has an infinite number of terms; (2) the infinite decimal expansion has a *last* element at the "infinitieth place," and since the latter is "infinitely far out" in the decimal expansion, this element is infinitely small or infinitesimal in comparison to finite magnitudes; and (3) since continuity requires irrationals, continuity presupposes infinitesimals. Furthermore, the method of defining irrational points by nested intervals[35] was *misconstrued* by du Bois-

[32] For details on the definition of "measure" for various kinds of point-sets, the reader is referred to H. Cramér, *Mathematical Methods of Statistics* [134], pp. 22*ff.*, and P. R. Halmos, *Measure Theory* [135].
[33] H. Weyl, *Philosophy of Mathematics and Natural Science* [130], p. 44.
[34] C. Hartshorne and P. Weiss (eds.), *The Collected Papers of Charles Sanders Peirce*, Vol. VI (Cambridge: Harvard University Press, 1935), paragraph 125.
[35] R. Courant and H. Robbins, *What is Mathematics?* [133], pp. 68–69.

Reymond[36] such that he was then able to charge it with committing the Bernoullian fallacy.[37]

We are now concerned with this fallacy, because it is always committed when the attempt is made to use the *infinite divisibility* of positive intervals as a basis for deducing Zeno's metrical paradox and for then denying that a positive interval can be an infinitely divisible extension. Precisely this kind of deduction of the paradox is attributed to Zeno by H. D. P. Lee[38] and P. Tannery,[39] both of whom seem to be unaware of the fallacy involved.

The following basic assumptions are involved in their version of Zeno's arguments:

1) Infinite *divisibility* guarantees the possibility of a *completable* process of "infinite division," i.e., of a completable infinite sequence of sets of division operations.

2) The completion of this process of "infinite division" is achieved by the last set of division operations in the sequence and terminates in "reaching" a last product of division in each of the parts—a mathematical point of zero extension.[40]

[36] P. du Bois-Reymond, *Die Allgemeine Funktionentheorie*, Vol. I, pp. 58–67.

[37] Du Bois-Reymond's fundamental error lies in supposing that the method of nested intervals allows and requires the "coalescing" of the end-points of a supposedly "next-to-the-last" interval into a single point such that this "coalescing" is the *last step* in an infinite progression of nested interval formations. If the method in question did require such a coalescing, then it would indeed be as objectionable logically as is the Bernoullian conception of the ∞th or last natural number. This is not the case, however, for while the method does indeed make reference to a progression of intervals, it neither allows nor requires that the irrational point is the "last" or "∞th" such "contracted" interval. Instead of appealing to "coalescence," the method specifies the irrational point *by the mode of variation* of the intervals in the *entire sequence*. It is therefore a property of the entire sequence which enables us to define the kind of point which is being asserted to exist. It would seem that du Bois-Reymond permitted himself to be misled by such pictorial language as "The interval contracts into a point."

[38] H. D. P. Lee, *Zeno of Elea* [2], p. 23.

[39] P. Tannery, "Le Concept Scientifique du Continu: Zénon d'Élée et Georg Cantor," *Revue Philosophique*, XX, No. 2 (1885), pp. 391–392.

[40] This assumption is to be likened to the supposition that the printing of all the $\aleph_0$ digits in the infinite decimal representation of π would be completed by printing a *last* digit; cf. the discussion in Chapter II, §4 [pp. 222–226 below].

3) The *actual infinity* of distinct point-elements constituting the interval is generated by such an alleged process of "infinite division."

4) Since the sets of divisions begin with a first operation on the total interval, each has an immediate successor, and each set, except the first, has a specific predecessor, they jointly constitute a progression of sets of one or more operations.

By assumptions (3) and (4), the "final elements" or points of the interval to which Zeno's metrical argument is to be applied are each presumed to have been generated by the last step in a *progression of division operations*. This consequence, however, is absurd. For it is the very essence of a progression not to have a last term and not to be completable in that ordinal sense! To maintain the self-contradictory proposition that in such an actually infinite aggregate of order type ω, there is a "last" set of divisions which ensures the completability of the process of "infinite division" by "reaching" a "final" product of division is indeed to commit the Bernoullian fallacy.

Several consequences follow at once:

1) We do not ever "arrive" by this kind of "infinite division" of an interval at its actual, super-denumerable infinity of mathematical points in the sense of first *generating* this actual infinity of unextended elements by "infinite division."

2) *The facts of infinite* (i.e., indefinite) *divisibility do not by themselves legitimately give rise to the metrical paradoxes of Zeno*, which *may* arise if we postulate an actual infinity of point-elements *ab initio*. It is because Cantor's theory rests on this latter postulate and not because every interval on his number axis is infinitely (i.e., indefinitely) divisible that we must inquire whether the line as conceived by Cantor is beset by the metrical difficulties pointed out by Zeno.

To show that this latter assertion is justified within the context of point-set theory, we shall now construct on the foundations of that theory a treatment of infinite divisibility consistent with it.

No clear meaning can be assigned to the "division" of a line unless we specify whether we understand by "line" an entity like a sensed "continuous" chalk mark on the blackboard or the very differently continuous line of Cantor's theory. The "continuity" of the *sensed* linear expanse consists essentially in its failure to exhibit visually noticeable gaps as the eye scans it from one of its extremities to the other. There are no distinct *elements* in the sensed "continuum" of which the seen line presents itself as a structured aggregate. By contrast, the continuity of the Cantorean line consists precisely in the complicated structural relatedness of (point) *elements* which is specified by the postulates for real numbers.[41]

We cannot always perceive a distinct third gap between any two visually discernible gaps (sections) in the sensed line. Thus the visually discernible gaps (sections) in that line do not constitute a discernibly dense set. This means that any significant assertion concerning possible divisibility of a *sensed* line must be compatible with the existence of thresholds of perception. Division of the *sensed* line will mean the creation of one or more perceptible gaps in it. Contrariwise, any attribution of (infinite) "divisibility" to a Cantorean line must be based on the fact that *ab initio* that line and its intervals are already "divided" into an actual dense infinity of point-elements of which the line (interval) is the structured aggregate. Accordingly, the Cantorean line can be said to be already actually *infinitely divided*. "Division" of the line can therefore mean neither the creation of visual gaps in it nor the "separation" of the point-elements from one another to make them distinct. *What we will mean in speaking of the "division" of the Cantorean line is the singling out of positive non-overlapping subintervals from (proper or improper) intervals of the line,* and in the case of finite point-sets in general and of the degenerate interval in particular, "division" will mean the formation of proper non-

[41] See the earlier discussion in [*Modern Science and Zeno's Paradoxes*] Chapter II, end of §2A.

empty *subsets*. A positive interval is *infinitely* divisible in the sense of permitting the SINGLING OUT of at least one *denumerable* infinity of positive, non-overlapping subintervals.

It follows from our definition of division and from the properties of finite sets that the division of a *finite* point-set of two or more members necessarily effects a reduction in its cardinality. This reduction is in marked contrast to the behavior of *intervals,* whose division yields subintervals of the same cardinality as the original interval. It is of fundamental importance to be aware in this context that the division of an interval effects *no* reduction in the cardinality of the resulting subintervals as compared to that of the original interval. For the unwitting *denial* of this fact seems to be implicit (along with the Bernoullian fallacy) in the false supposition that the infinite divisibility of an interval assures the obtainability of all of its constituent individual points as "products of infinite division." Since the degenerate interval has no proper non-empty subset, that unique kind of interval is *indivisible*. We see that on our theory, (infinite) divisibility and indivisibility are respectively *set-theoretic* rather than metrical properties. This theory has enabled us to assign a precise meaning to the indivisibility of a unit point-set by (1) defining division as an operation on sets only and not on their elements, (2) defining divisibility of finite sets as the formation of proper non-empty subsets of these, and (3) showing that the degenerate interval is indivisible by virtue of its lack of a subset of the required kind.

Note that division is a kind of *operation* on specified point-sets while divisibility and being super-denumerably infinite are respective *properties* of certain point-sets in the case of the Cantorean line. And the infinite divisibility of an interval does not make for a kind of "infinite division" which would first generate its super-denumerably many constituent points.[42]

It is of importance to realize that our analysis has shown

[42] Nevertheless, it is often convenient *by way of elliptic parlance* to designate the membership of a set through mention of an actual infinity of operations which, as it were, "identify" the elements of the set in question.

how we can assert the following *two* propositions *perfectly consistently:*

1) The line and positive intervals in it are *infinitely divisible.*

2) The line and positive intervals in it are each a union of *indivisible* degenerate intervals.

We are now prepared to deal with the crux of our problem by using point-set theory to refute the second horn of Zeno's metrical dilemma.

§4. THE STATUS OF ZENO'S PARADOX OF EXTENSION

Since a positive interval is the union of a continuum of degenerate intervals,[43] we must now determine what meaning, *if any,* we can assign to "summing" the lengths of all these degenerate intervals with a view to obtaining the paradoxical value zero for the length of the total interval. The answer we shall give to this problem will not be *ad hoc,* since the reasoning on which it is based will not depend upon the particular lengths which Zenonians wish us to "compound" but rather on the fact that the number of lengths to be "added" is *not denumerable.*

Earlier, we determined the length of the union of a *finite* number of non-overlapping intervals of known lengths on the basis of these latter lengths. In addition, we made a corresponding determination of the length of the union of a *denumerable* infinity of non-overlapping intervals. If we now attempt to subdivide an interval into a *non-denumerable* infinity of non-overlapping intervals, we find that they *cannot be non-degenerate.* For Cantor has shown that any collection of positive non-overlapping intervals on a line is at most denumerably infinite.[44] It follows that the degenerate subintervals which are at the focus of our interest are the only kind

[43] The word "continuum" can designate either the ordering structure of the real numbers or their cardinality. The context will indicate which of these meanings is intended or whether both are jointly involved.

[44] G. Cantor, *Gesammelte Abhandlungen,* p. 153.

of non-overlapping subintervals of which there are non-denumerably many in a given interval. Quite naturally, therefore, they create a special situation. The latter is due to the fact that our theory does *not* assign any meaning to "forming the arithmetic sum," when we are attempting to "sum" a *super-denumerable* infinity of individual numbers (lengths)! This fact is independent of whether the individual numbers in such a non-denumerable set of numbers are zeros or finite cardinal numbers differing from zero.

Consequently, the theory under discussion cannot be deemed to be *ad hoc* for precluding the possibility of "adding," in Zenonian fashion, the zero lengths of the continuum of points which "compose" the interval (a, b) to obtain zero as the length of this interval. Though the finite interval (a, b) is the union of a continuum of degenerate subintervals, *we cannot meaningfully determine its length in our theory by "adding" the individual zero lengths of the degenerate subintervals.* We are here confronted with an instance in which set-theoretic addition (i.e., forming the union of degenerate subintervals) is meaningful while arithmetic addition (of their lengths) is not.

We have shown that the standard set-theoretical geometry here presented does not have the paradoxical feature of both assigning the non-zero length $b - a$ to the interval (a, b) *and* permitting the inference via the additivity of lengths that (a, b) must have zero length on the grounds that its points each have zero length. More precisely, we have shown that geometrical theory can consistently affirm the following four propositions simultaneously in the context of its rules of additivity for lengths:

1) The finite interval (a, b) is the union of a continuum of degenerate subintervals.

2) The length of each degenerate (sub)interval is 0.

3) The length of the interval (a, b) is given by the number $b - a$.

4) The length of an interval is not a function of its cardinality.

Our analysis has manifestly refuted the Zenonian allegation of inconsistency if made against the standard set-theoretical geometry.

The set-theoretical analysis of the various issues raised or suggested by Zeno's paradoxes of plurality has enabled me to give a *consistent* metrical account of an extended line segment as an aggregate of unextended points. Thus Zeno's mathematical paradoxes are avoided in the formal part of a geometry or chronometry built on Cantorean foundations. Given the aforementioned additivity rules for length of the standard mathematical theory, the consistency of the metrical analysis which I have given requires the *non-denumerability* of the infinite point-sets constituting the intervals on the line. Thus, if any infinite set of *rational* points were regarded as constituting an extended line segment, then the customary mathematical theory under consideration could assert the length of that merely denumerable point-set to be greater than zero only at the cost of permitting itself to become self-contradictory! For we saw that in the standard theory the length of an interval and the measure of a point-set are each countably additive. And hence if an interval (a, b) between the *rational* points a and b *were* claimed to consist *only* of the denumerable *rational* points between a and b, the following logical situation would result: The denumeration of this set of points coupled with the countable additivity of their zero lengths would permit the deduction that the length of (a, b) is (paradoxically) zero. This zero result is deducible without any reference at all to the congruences and unit of length furnished by a transported standard of length, which is extrinsic to (a, b). To emphasize the independence of this result from a length-standard extrinsic to (a, b), we can say that the *"intrinsic"* length of a denumerable "interval" of rational points is zero—similarly for the measure of such an "interval."[45]

It might seem that this conclusion concerning the fundamental logical importance of non-denumerability could be criticized in the following way: The need for non-denumerably infinite point-sets to avoid metrical contradictions derives from the countable additivity of length and measure. Without these additivity rules, it would not have been possible to infer that the length and the measure of an enumerable point-set turn out to be *zero*. Consequently, by omitting these additivity rules, it would presumably have been possible to assign a finite length to certain enumerable sets without contradiction and to base physical theory on a denumerable geometry. Thus it might be argued that a non-denumerably infinite point-set is only unimportantly indispensable for consistency, since this indispensablity obtains only relatively to a formulation of the theory in which length and measure are countably additive.

To assess the merits of this objection, two points must first be noted:

1) The rejection of *countable* additivity for length and measure would entail incurring the loss of those portions of standard applied mathematics which depend on the presence of countable additivity in the foundations. Thus, for example, one would need to sacrifice some of the mathematics of Fourier series and of the eigenfunctions of quantum mechanics as well as of probability theory and statistics. For countably additive set functions enter into these branches of applied mathematics in one or another form via the Lebesgue integral, the Lebesgue measure, or the Lebesgue-Stieltjes integral.

2) Apart from being required for metrical consistency in the context of countable additivity, the super-denumerability of intervals is inherent in the assumption of the mathematical continuity of space and time and thus in everything that depends on this assumption in the theories of empirical science. Those who maintain that super-denumerably infinite point-sets are only quite unimportantly essential to physical theory are making a gratuitous claim and have so far given us nothing more than the *recommendation* to *attempt* to

[45] Cf. also H. Cramér, *Mathematical Methods of Statistics* [134], p. 25.

erect the physics of space and time on *denumerable* foundations. For to substantiate their claim, they must demonstrate that the implementation of their recommendation is feasible by showing the following: At least one kind of mathematics which avoids Zeno's paradox by dispensing with countable additivity in the interest of postulating the denumerability of space and of time is fully as viable for empirical science as the standard mathematics used in actual physical theory.[46] But in the light of the physical considerations put forward in favor of countable additivity in [72] Chapter II, §2A, it is quite doubtful that physicists would acquiesce in its sacrifice. In this significant sense, Zeno's metrical paradox of extension does pose a challenge to theorists whose philosophical commitments do not allow them to avail themselves of super-denumerably infinite sets.

Proponents of Zeno's view might still argue that this *arithmetical* rebuttal, which appeals to the fact that arithmetic addition is not defined for a super-denumerable infinity of numbers, is unconvincing on purely geometric grounds, maintaining that if extension (space) is to be composed of elements, these must themselves be extended. Specifically, geometers such as Veronese objected[47] to Cantor that in the array of points on the line, their extensions are all, as it were, "summed geometrically" before us. And from this geometric perspective, it is not cogent, in their view, to suppose that even a super-denumerable infinity of unextended points would be able to sustain a positive interval, especially since the Cantorean theory can claim arithmetical consistency here only because of the obscurities that obligingly surround the meaning of the arithmetic "sum" of a super-denumerable infinity of numbers.

Is this objection to Cantor conclusive? I think not. Whence

[46] For doubts about the thesis that the warrant for the mathematical continuity of space and time is conventional rather than empirical, cf. my *Philosophical Problems of Space and Time* [139], pp. 334–336.

[47] See E. W. Hobson, *The Theory of Functions of a Real Variable*, 2nd edition, Vol. I (New York: Cambridge University Press, 1921), pp. 56–57.

does it derive its plausibility? It would seem that it achieves persuasiveness via a tacit appeal to a *pictorial* representation of the points of mathematical physics in which they are arrayed in the consecutive manner of beads on a string to form a line. But the properties that any such representation imaginatively attributes to points are not even allowed, let alone prescribed, by the formal postulates of geometric theory. The spuriousness of the difficulties adduced against the Cantorean conception of the line becomes apparent upon noting that not only the cardinality of its constituent points altogether eludes pictorialization but also their dense ordering: between any two points, there is an infinitude of others. Thus, in complete contrast to the discrete order of the beads on a string, *no* point is immediately adjacent to any other. The futility, irrelevance, and misleading effect of attempts to visualize the Cantorean interval structurally become apparent from the following: If we were to exclude one end-point of an initially closed interval from that interval, the now open "end" of that interval would *defy* pictorialization because of the non-existence of a point *next to* the excluded point.

These considerations show that from a genuinely geometric point of view, a physical interpretation of the formal postulates of geometry cannot be obtained by the inevitably misleading pictorialization of *individual* points of the theory. Instead, we can provide a physical interpretation quite unencumbered by the intrusion of the irrelevancies of visual space, if we associate *not* the term *"point"* but the term *"linear continuum of points"* of our theory with an appropriate body in nature. By a point of this body we then mean nothing more or less than an element of it possessing the formal properties prescribed for points by the postulates of geometry. And, on this interpretation, the ground is then cut from under the geometric *parti pris* against Cantor by the modern legatees of Zeno.

It has been overlooked in some quarters that the issues posed by Zeno's paradox of extension are no less important philosophically than are those raised by his paradoxes of motion. Two examples will illustrate that there has been insufficient appreciation of the philosophical lesson to be learned from the avoidance of Zeno's paradox of extension within the framework of the standard mathematical theory.

1) In his discussion of the mathematical theory of motion, Russell neglected the essential contribution made by the cardinality and ordinal structure of the linear Cantorean continuum toward the avoidance of Zeno's paradox of extension. This philosophical neglect of his is clear in the following passages:

> Mathematicians have distinguished different degrees of continuity, and have confined the word "continuous," for technical purposes, to series having a certain high degree of continuity. But for *philosophical* purposes, all that is important in continuity is introduced by the lowest degree of continuity, which is called "compactness" [i.e., denseness]. . . . What do we mean by saying that the motion is continuous? It is not necessary for our purposes to consider the whole of what the mathematician means by this statement: *Only part of what he means is philosophically important.* One part of what he means is that, if we consider any two positions of the speck occupied at any two instants, there will be other intermediate positions occupied at intermediate instants. . . .[48]

We know that the mere existence of the denseness property guarantees only a denumerably infinite point-set. But in the context of the standard mathematical additivity rules for length, a super-denumerably infinite point-set is required by the demands of metrical consistency. And it could be reasonably maintained that the *physical relevance* of the metrical concept of length requires its countable additivity. Hence in this sense there are *philosophical* reasons for requiring a higher degree of continuity than is ensured by the denseness property alone.

2) The Greeks certainly were *not* led to the discovery of

[48] B. Russell, *Our Knowledge of the External World* [100], pp. 144, 146, my italics.

incommensurable magnitudes by merely *operationally* carrying out the iterative transport of measuring sticks.[49] And it is impossible to show by direct physical operations alone that there are hypotenuses whose length cannot be represented by any rational number. For the limits of experimental accuracy and the denseness of the rational points guarantee that we can never claim anything but a rational result on the strength of operational accuracy alone. A radical operationist approach to geometry might therefore suggest that this science be constructed so as to use only the system of rational points.[50] The analysis given in this chapter has aimed to show that in the absence of a denumerable alternative to the standard mathematical theory which is demonstrably viable for the purposes of physics, such an operationist approach to geometry and to the theoretical measurables of physics must be rejected on logical grounds.[51]

[49] For the historical details, see K. von Fritz, "The Discovery of Incommensurability by Hippasus of Metapontum," *Annals of Mathematics*, XLVI (1945).

[50] Cf. the approximative geometry of J. Hjelmslev ("Die natürliche Geometrie," *Abhandlungen aus dem mathematischen Seminar der Hamburger Universität*, Vol. II [1923], pp. 1*ff.*) and Weyl's comments on it (H. Weyl, *Philosophy of Mathematics and Natural Science* [130], pp. 143–144).

[51] In §3 (pp. 336–338), G. J. Massey, "Toward a Clarification of Grünbaum's Conception of an Intrinsic Metric," *Philosophy of Science*, XXXVI (1969), pp. 331–345, offers some criticisms of the formulation of the thesis of this chapter. For a discussion of these criticisms, see A. Grünbaum, "Reply to Critiques, and Critical Exposition," *Philosophy of Science*, XXXVII (1970), forthcoming.