From the Calculus to Set Theory, 1630–1910

An Introductory History

Edited and with an introduction by

I. Grattan-Guinness

with chapters by

H. J. M. Bos
R. Bunn
J. W. Dauben
I. Grattan-Guinness
T. W. Hawkins
K. Møller Pedersen

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earlier mathematicians by present mathematical standards, nor to emphasise the inadequacy of their concepts as compared to modern ones. On the contrary, a historian of mathematics ought to enter into the mode of thought of the period under consideration in order to bring out the development of the mathematical ideas in its historical context. Briefly, it may be said that the mathematicians in the period preceding the invention of the calculus blazed the trail for its invention. They did so by employing heuristic methods, by making the geometry analytical, and by seeking methods for solving problems of quadratures and tangents.[1]

[1] I am grateful to Dr. John North of Oxford University for correcting some of my linguistic mistakes, and to Dr. D. T. Whiteside of Cambridge University for his valuable comments on the manuscript.

Chapter 2

Newton, Leibniz and the Leibnizian Tradition

H. J. M. Bos

2.1. *Introduction and biographical summary*

The starting-point of this chapter is the ' invention ', or rather ' inventions ', of the calculus. Both Newton (in 1664–1666) and Leibniz (in 1675) created, independently of each other, an infinitesimal calculus. Their inventions were very different in concepts and style, but each contains so much of what we now recognise as essential to the calculus that the expression ' invention of the calculus ' is justified in both cases. I go on to consider the subsequent development of the calculus till about 1780. In this development the Leibnizian type of calculus with differentials and integrals proved more successful than the Newtonian fluxional calculus ; therefore I concentrate on the former.

Many great and lesser mathematicians were involved in the development of the calculus in the period covered by this chapter. I shall restrict myself to those who played the prime roles in the story : Isaac Newton, Lucasian professor of mathematics at Cambridge and later Master of the Mint in London ; Gottfried Wilhelm Leibniz, historian and scientist at the ducal court of Hanover ; Jakob Bernoulli, professor of mathematics at Basle ; his brother Johann Bernoulli, younger by thirteen years, who after a professorate at Groningen succeeded Jakob in Basle in 1705 ; Guillaume François Marquis de l'Hôpital, a French nobleman living by private means, and an able mathematician eagerly interested in the new developments in infinitesimal methods ; and finally Leonhard Euler, who studied with Johann Bernoulli and then entered a career in the typically 18th-century scientific institutions, the academies. He was professor at the St. Petersburg (now Leningrad) Academy from 1730 to 1741 and from 1766 till his death ; in the intervening years he served the Berlin Academy as professor.

Many of the great ideas that were to make Isaac Newton famous in mathematics and natural science came to him in the years 1664–1666.

At that time he was a graduate student at Trinity College, Cambridge, but for some time during those two years he lived in Lincolnshire, staying away from Cambridge for fear of the Plague (compare Whiteside *1966a*). His ideas on gravity, which he was to work out later and present to the world in his famous *Principia* (*1687a*), date from that period, as well as his theory of colours, published in the treatise *Opticks* in 1704, the binomial series theorem and his fluxional calculus, which we shall discuss in more detail in section 2.2.

As with gravity and colours, publication of these mathematical ideas in print was long delayed. Newton did compose several accounts of his findings in infinitesimal calculus. In October 1666 he summarised the discoveries of the fruitful two years in a tract on fluxions (*1666a*); in 1669 he wrote a treatise on infinite series, the *De analysi* (*1669a*), which circulated in manuscript form among members of the Royal Society; from 1671 dates a treatise on the method of fluxions and infinite series (*1671a*); and in about 1693 he composed a treatise on the quadrature of curves (*1693a*). However, the 1666 tract and the treatise on the method of fluxions were not published in his lifetime, the *De analysi* was published only in 1711, and the treatise on quadratures of curves in 1704. Meanwhile the *Principia* of 1687 had brought for the first time to the general public indications of his methods in infinitesimal calculus, but these were not enough to show the scope and power of his mathematical discoveries.

About the turn of the century a fair amount was published about Leibniz's calculus (as we shall see in sections 2.5–2.8 below), and sufficient information about Newton's calculus was available to show that both men had found new methods in essentially the same mathematical field. This caused a nasty quarrel over priority, in which feelings of personal and national pride combined with insufficient insight in the mathematics involved (at least in the case of the lesser participants in the debate) to create a distasteful muddle of misunderstandings and insinuations which has only been cleared up through patient historical research in the present century. The net result of the historical research is that Leibniz found his calculus later than Newton and independently of him, and that he published it earlier.

In 1669 Newton had succeeded Isaac Barrow as Lucasian professor, but in the 1690s he grew dissatisfied with his position at Cambridge. He visited London often, to attend meetings of the Royal Society, of which he was a fellow from 1672, and to be present at sessions of Parliament as a member for the Cambridge University constituency. He moved finally to London in 1696 when he was offered the office of Warden of the Mint. In 1703 he became president of the Royal Society, a post which he held till his death. His position as the most eminent British scientist was further emphasised by a knighthood in 1705.

By the 1710s so much on the fluxional calculus was in print that the method was taken up and applied by others. However, this further development of the Newtonian type of calculus remained restricted to Great Britain, and it did not achieve much. Reasons of the lack of success lie in the isolation from the Continental developments in analysis because of the priority dispute, in the lack of mathematicians in Britain of sufficient stature to really develop Newton's calculus, and in an overstressed loyalty to Newton's conception of the calculus and to his notations, which were less versatile than Leibniz's.

On the Continent Leibniz's inventions gave rise to a much more intense development, to whose origins in the 1670s we now turn.

Before Leibniz entered the service of the house of Hanover in 1676 he had spent four years in Paris on a diplomatic mission, which left him ample time to pursue his interest in mathematics, the sciences, history, philosophy and many other things. He met many French philosophers and made two visits to London to the Royal Society. The Paris years were his formative period. When he arrived in 1672 his knowledge of mathematics was slight, despite the fact that he had published a small tract on combinatorics. He was trained in law at the university of his home town of Leipzig. In Paris Christiaan Huygens, who lived there at that time, recognised Leibniz's mathematical abilities and guided his first studies in the higher mathematics. Leibniz's 'growth to mathematical maturity' (see Hofmann *1949a*) was indeed impressive; it led to his discovery of the calculus in 1675, the elaboration of that calculus in the following years and its publication in 1684–1686. He contributed to other branches of mathematics as well, for instance to algebra (solvability of equations, determinants) and to nearly all other fields of human learning, including religion, politics, history, physics, mechanics, technology, mathematics, geology, linguistics and natural history. Many of his results were not immediately published and became known only gradually, through correspondence (from his comparative intellectual isolation in Hanover Leibniz corresponded with over a thousand scholars), through publication of short articles in journals (he was one of the founders of the first scientific journal in Germany, the *Acta eruditorum*), and later through the publication of his manuscripts, most of which he kept and which are now stored at the Leibniz archive in Hanover.

Leibniz's publication of his calculus in two articles in the *Acta* of 1684 and 1686 did not provoke great commotion in mathematical circles. The articles were rather short, and they were marred by misprints and in places deliberately obscure, so that it is in fact surprising that in the following decade they were understood at all.

Jakob and Johann Bernoulli studied the articles from 1687, and by 1690 they showed, in articles published in the *Acta*, that they had mastered the Leibnizian symbolism and its use. They both started a correspondence with Leibniz ; the contact between Johann and Leibniz was especially intensive and productive. After 1690 a stream of articles in the *Acta* and in other journals, written by the Bernoullis and Leibniz and later joined by l'Hôpital and others, showed the learned world that the new calculus was something to be reckoned with.

However, for people of lesser mathematical calibre than the Bernoullis, it would have been very difficult actually to learn the calculus from these articles. What was wanted was a proper textbook of the calculus. Such a textbook came, though only of the differential calculus, in 1696 with l'Hôpital's *Analyse des infiniment petits pour l'intelligence des lignes courbes* ('Analysis of infinitely small quantities for the understanding of curved lines' : *1696a*).

The Marquis de l'Hôpital was introduced to the calculus by Johann Bernoulli, who, after finishing his medical studies in 1690, had travelled to Paris, where he impressed learned circles by a method to determine, by means of differentials, the curvature of arbitrary curves—a problem which by the methods of Cartesian analytic geometry was well nigh unsolvable. l'Hôpital was most impressed and asked Bernoulli to give him, for a good fee, lectures on the new method. Bernoulli accepted and the lectures were given, in Paris and at the country chateau of the Marquis. They were written out and both men kept copies. After about a year Bernoulli left Paris but agreed to continue instructing l'Hôpital by letter. In fact the agreement was that Bernoulli, for a handsome monthly salary, would answer all l'Hôpital's questions concerning mathematics, would send him all his mathematical discoveries and would give no one else access to these findings (see Bernoulli *Correspondence*, 144) ; a most curious and hardly honourable agreement which put Bernoulli's originality strictly in l'Hôpital's service. From the start Bernoulli did not quite keep to the letter of the contract, and l'Hôpital soon realised that he could not bind a brilliant mathematician in this way. But when in 1696 l'Hôpital published his textbook, and Bernoulli saw that most of its content was taken from his lectures with not more than a passing reference to the Marquis's indebtedness to Bernoulli, he could only be angry in silence, being bound by the contract.

Later, after l'Hôpital's death, Johann Bernoulli did try to get his part in the *Analyse* acknowledged, but by that time his credibility in priority questions had become very low because of open quarrels on such matters with his brother. Jakob Bernoulli was a rather introverted personality, but he was sensitive to praise from members of the mathematical community and he resented being overshadowed by his brilliant younger brother. Johann, on the other hand, liked his own success too much to spare his brother's feelings. So there appeared insinuating remarks in articles, and later a quarrel exploded and went on quite openly. Johann Bernoulli's claim to much of the content of the *Analyse* was found to be justified only when in 1921 the manuscript of his Paris lectures on the differential calculus was found (see Johann Bernoulli *1924a*).

However strained their mutual relations, through the writings of these men the Leibnizian calculus became known and proved its power. By the first decade of the 18th century other mathematicians devoted themselves to the new calculus, such as Jakob Hermann, Pierre Varignon, Niklaus Bernoulli (a nephew) and Daniel Bernoulli (son of Johann). The family Bernoulli continued to yield famous mathematicians throughout the 18th century.

In these early days the new calculus consisted mainly of rather loosely connected methods, and problems solved by these methods. The man who reshaped the Leibnizian calculus into a soundly organised body of mathematical knowledge was Leonhard Euler. Euler was the central figure of continental mathematics in the middle years of the 18th century. He published an enormous number of books and articles on mathematics, mechanics, optics, astronomy, navigation, hydrodynamics, technical matters such as artillery and shipbuilding, and very many other topics. He maintained this impressive productivity despite losing the sight of one eye in 1735 and becoming completely blind in 1766. His position at the academies involved him in many other tasks besides scientific research, such as advice on the performance of new inventions as fire-engines and pumps, and on technological enterprises like canal-building and the construction of water-works in the park of the royal palace *Sans Souci* of Prussia's Frederick the Great.

Euler's greatest influence on the calculus and on analysis in general was through his great textbooks, in which he gave analysis a definitive form, which it was to keep until well into the 19th century. These textbooks, written in Latin, were : *Introductio ad analysin infinitorum* ('Introduction to the analysis of infinites' : *1748a*), *Institutiones calculi differentialis* ('Textbooks on the differential calculus' : *1755b*), and *Institutiones calculi integralis* ('Textbooks on the integral calculus' : *1768–1770a*).

These were the men who created the calculus and shaped the Leibnizian tradition in analysis. In sections 2.3–2.8 I shall describe the mathematics involved, but first I shall devote the next section to an overview of the Newtonian calculus.

2.2. *Newton's fluxional calculus*

As was mentioned above, Newton's main mathematical discoveries in the infinitesimal calculus date from 1664 to 1666. (For a detailed account of his achievements in this period, see Newton *Papers*, vol. 1, 145–154, and *Works*$_2$, vol. 1, viii–xiii.) Autodidactically he quickly acquired adequate knowledge of existing theories in the field, benefitting especially from reading Descartes's *La géométrie* in van Schooten's edition with commentaries, and from the works of Wallis. Starting from these studies he developed in these fruitful two years his *fluxional calculus*.

In Newton's discoveries, complex, deep and many-sided as they are, a number of central themes may be distinguished. These are: series expansions, algorithms, the inverse relationship of differentiation and integration, the conception of variables as moving in time, and the doctrine of prime and ultimate ratios. Although these themes are interconnectedly present in almost all of his studies in the infinitesimal calculus, I shall deal with them separately.

Newton valued *power-series expansions* very highly, because they provide a means to reduce the analytical formulae of curves to a form in which all terms simply consist of a constant times a power of the variable. Thus transcendental curves (admitting no algebraic equation), as well as algebraic curves with complicated equations, can be represented by much simpler equations (be it with an infinite number of terms). Newton saw that this has two great advantages. Firstly, series expansion makes it possible to apply rules and algorithms which are defined for simple equations only, to a much wider range of curves. In particular, the relation

$$\int x^n\,dx=\frac{1}{n+1}\,x^{n+1}, \qquad (2.2.1)$$

which was known in various forms by the 1660s (see sections 1.10 and 1.11) can be used, in combination with power-series expansions, to provide series expressions for the quadratures of almost all curves. Secondly, series expansion provides a ready means for the approximation and simplification of formulae through the discarding of higher-order terms—a feature which he used with virtuosity in his applications of his mathematical methods to physical problems.

Newton's most famous series expansion is the 'binomial theorem', which he found in the winter of 1664–1665 and which states that the well-known binomial expansion for integer powers n,

$$(a+x)^n=a^n+\frac{n}{1}\,a^{n-1}x+\frac{n(n-1)}{1\,.\,2}\,a^{n-2}x^2+\ldots+x^n, \qquad (2.2.2)$$

can be generalised for fractional powers $\alpha=p/q$, in which case the right hand side of

$$(a+x)^\alpha=a^\alpha+\frac{\alpha}{1}\,a^{\alpha-1}x+\frac{\alpha(\alpha-1)}{1\,.\,2}\,a^{\alpha-2}x^2+\ldots \qquad (2.2.3)$$

is an infinite series. He found the theorem in connection with the problem of squaring the circle $y=(1-x^2)^{1/2}$. He compared the formulae $(1-x^2)^0$, $(1-x^2)^{1/2}$, $(1-x^2)^{2/2}$, $(1-x^2)^{3/2}$, $(1-x^2)^{4/2}$, ... The first, third, fifth, ... formulae involve no root, and therefore the quadratures of the corresponding curves are easily found:

$$\left.\begin{aligned}&\text{quadrature of } y=(1-x^2)^0 \text{ is } x,\\&\text{quadrature of } y=(1-x^2)^{2/2} \text{ is } x-\tfrac{1}{3}x^3,\\&\text{quadrature of } y=(1-x^2)^{4/2} \text{ is } x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5.\end{aligned}\right\} \qquad (2.2.4)$$

On examining the coefficients in these expansions, Newton noted that the denominators are the odd numbers 1, 3, 5, 7, ... and that the numerators are, in the successive expansions, {1}, {1, 1}, {1, 2, 1}, {1, 3, 3, 1}, ..., that is, the numbers in the 'Pascal triangle', which he knew could be expressed for successive integral values of n as

$$\left\{1,\ n,\ \frac{n(n-1)}{1\,.\,2},\ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3},\ \ldots\right\}.$$

He then guessed that, by analogy, the same expressions would apply for *fractional* values of n. When $n=\frac{1}{2}$ this yields:
quadrature of $y=(1-x^2)^{1/2}$ is

$$x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{8}x^5}{5}-\frac{\frac{1}{16}x^7}{7}-\frac{\frac{5}{128}x^9}{9}-\ldots. \qquad (2.2.5)$$

He then saw that this procedure of guessing, or 'interpolating', expansions such as (2.2.5) from the scheme of the series (2.2.4) could be applied to the equations of the curves as well as to their quadratures, and in this way he found that

$$(1-x^2)^{1/2}=1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^5-\tfrac{1}{16}x^6-\tfrac{5}{128}x^8-\ldots. \qquad (2.2.6)$$

Not satisfied with the reliability of the interpolation procedure, he checked (2.2.6) in two ways. He showed that the product of the right hand side of (2.2.6) with itself yields $1-x^2$ (that is, all further coefficients in the product series are zero), and he saw that a common method of root extraction known as the 'galley method', applied formally to $1-x^2$, yields the same series. In the same way as with root extraction, he used the algorithm of long division to obtain series expansions, for instance,

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-\ldots, \qquad (2.2.7)$$

which provided the quadrature of the hyperbola $y=1/(1+x)$. He also obtained (2.2.7) by assuming that the binomial expansion applied when $n=-1$.

In the *De analysi* (*1669a*), in which these methods of series expansions are explained and used, Newton also provides a general rule to compute, for a given polynomial equation

$$\sum a_{ij}x^iy^j=0 \qquad (2.2.8)$$

between x and y, the first coefficients of the pertaining series

$$y=\sum b_ix^i \qquad (2.2.9)$$

(*Papers*, vol. 2, 222–247).

Both in the way that Newton found the binomial theorem and in the application of series expansions in general, the relation, which we now write as

$$\int x^n\,dx=\frac{1}{n+1}x^{n+1}, \qquad (2.2.10)$$

plays an important role. He mentioned this 'quadrature of simple curves' at the outset of his *De analysi*: 'RULE 1. If $ax^{m/n}=y$, then will $(na/(m+n))x^{(m+n)/n}$ equal the area ABD' (*ibid.*, 206–207; see figure 2.2.1). Later in that treatise he gave a general procedure (of which rule 1 is a direct consequence) for finding the relation between the quadrature of a curve (as AD in figure 2.2.1) and its ordinate. The procedure makes it clear that Newton recognised the *inverse relationship of integration and differentiation* (although, of course, he did not use these terms). He explains his method by means of an example, from which, however, the generality of the procedure is quite clear. He

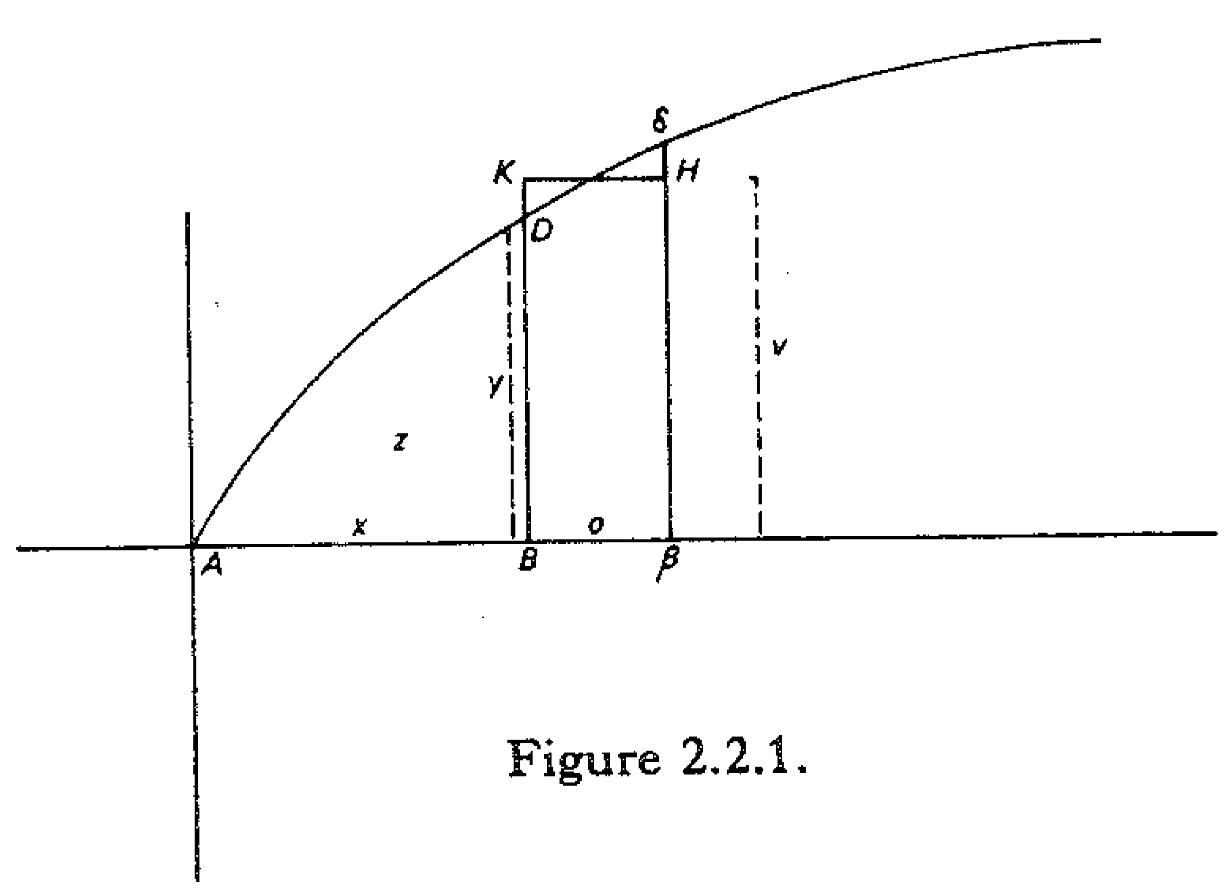

Figure 2.2.1.

proceeds as follows (*ibid.*, 242–245). In figure 2.2.1 let area $ABD=z$, $BD=y$ and $AB=x$; let further $B\beta=o$ and let $BK=v$ be chosen such that area $BD\delta\beta=$area $BKHb=ov$. Consider, as example, the curve for which

$$z=\tfrac{2}{3}x^{3/2}, \qquad (2.2.11)$$

that is (removing roots to get a polynomial equation),

$$z^2=\tfrac{4}{9}x^3; \qquad (2.2.12)$$

then also

$$(z+ov)^2=\tfrac{4}{9}(x+o)^3, \qquad (2.2.13)$$

from which

$$z^2+2zov+o^2v^2=\tfrac{4}{9}(x^3+3x^2o+3xo^2+o^3). \qquad (2.2.14)$$

Now by removing the terms without o, which are equal on both sides from (2.2.12), and dividing the remainder by o, we obtain

$$2zv+ov^2=\tfrac{4}{9}(3x^2+3xo+o^2). \qquad (2.2.15)$$

Now Newton takes $B\beta$ 'infinitely small', in which case, as the figure suggests, $v=y$ and the terms with o vanish:

$$2zy=\tfrac{4}{3}x^2. \qquad (2.2.16)$$

Inserting the value of z from (2.2.11), he obtains

$$y=x^{1/2}. \qquad (2.2.17)$$

Clearly the procedure is applicable to all polynomial relations between x and z. It consists in essence of calculating the derivative (in this case the y) for any algebraic function z of x.

Newton saw clearly that the problem of quadratures was to be approached in this inverse way: by calculating y for all manner of algebraic z, he could find all manner of curves (y, x) which are quadrable. Indeed, he calculated many such quadrable curves, writing them together in extensive lists, which are thus nothing less than the first tables of integrals (compare *Papers*, vol. 1, 404–411).

The essential element in the foregoing procedure is the substitution of 'small' corresponding increments o and ov for x and z in the equation. In studies on the determination of maxima and minima, tangents and curvature, Newton had extensively made use of this method, and he had worked out various *algorithms* for these problems, by which he could calculate the slope of the tangent or the curvature in any point of an algebraic curve. (In modern terms, he had developed algorithms to determine the derivative of any algebraic function.) Later he reformulated these algorithms and their proofs in terms of fluents and fluxions, and we shall come back to them after discussing these concepts.[1]

[1] Compare, for instance, Newton *1671a*, in *Papers*, vol. 3, 72–73.

The terms 'fluents' and 'fluxions' indicate Newton's conception of variable quantities in analytical geometry: he saw these as 'flowing quantities', that is, *quantities that change with respect to time*. Thus, when considering the curve of figure 2.2.1, he would conceive the point D as moving along the curve, while correspondingly the ordinate y, the abscissa x, the quadrature z or any other variable quantity connected with the curve would increase or decrease, or in general change or 'flow'. He called these flowing quantities 'fluents' (as opposed to the constant quantities occurring in the figure or in the problem at hand), and he called their rate of change with respect to time their 'fluxion'. In his earlier researches he indicated fluxions by separate letters; in *1671a* he introduced the dot-notation, where the fluxions of the fluents x, y, z are $\dot{x}, \dot{y}, \dot{z}$ respectively.

It should be remarked that the way in which the fluents vary with time is arbitrary. Newton often makes, for simplicity, an additional assumption about the movement of the variables, supposing that one of the variables, say x, moves uniformly, so that $\dot{x}=1$. Such assumptions can be made because the values of the fluxions themselves are not of interest but rather their ratio, such as $\dot{y}/\dot{x}$, which gives the slope of the tangent. By this conception of quantities moving in time Newton thought himself able to solve the foundational difficulties inherent in considering 'small' corresponding increments of variables, which are so small that we may discard them, and yet are not equal to zero, as we want to divide through by them. In his approach to this problem, his theory of ***prime and ultimate ratios***, which we shall discuss in section 2.10, his conception of flowing quantities is essential; through this conception he comes very near to a use of limits as foundation of the calculus.

We now return to the *algorithms* mentioned above. The corresponding increments of variables, can be expressed in terms of fluxions: let o now be an infinitesimal element of time, then the corresponding increments of the fluents $x, y, z, \ldots$ are $\dot{x}o, \dot{y}o, \dot{z}o, \ldots$ respectively. The ratio of $\dot{y}$ to $\dot{x}$ can now be determined in a way which is evident in the following example, which Newton gives himself in *1671a* (*Papers*, vol. 3, 79–81). Let a curve be given with equation

$$x^3-ax^2+axy-y^3=0. \qquad (2.2.18)$$

Substituting $x+\dot{x}o$ and $y+\dot{y}o$ for x and y respectively yields

$$\begin{aligned}(x^3+3\dot{x}ox^2+3\dot{x}^2o^2x+\dot{x}^3o^3)-(ax^2+2a\dot{x}ox+a\dot{x}^2o^2)\\ +(axy+a\dot{x}oy+a\dot{y}ox+a\dot{x}\dot{y}o^2)\\ -(y^3+3\dot{y}oy^2+3\dot{y}^2o^2y+\dot{y}^3o^3)=0. \qquad (2.2.19)\end{aligned}$$

Deleting $x^3-ax^2+axy-y^3$ as equal to zero from (2.2.18), dividing through by o and discarding the terms in which o is left, yields

$$3\dot{x}x^2-2a\dot{x}x+a\dot{x}y+a\dot{y}x-3\dot{y}y^2=0, \qquad (2.2.20)$$

from which the ratio of $\dot{y}$ and $\dot{x}$ is easily obtained:

$$\frac{\dot{y}}{\dot{x}}=\frac{3x^2-2ax+ay}{3y^2-ax}. \qquad (2.2.21)$$

We note that the numerator and the denominator in the result are (apart from a sign) the partial derivatives f_x and f_y of $f(x,y)=x^3-ax^2+axy-y^3$, the left hand side of the equation of the curve. Thus

$$\frac{\dot{y}}{\dot{x}}=-\frac{f_x}{f_y}. \qquad (2.2.22)$$

Indeed, this relation is implicit in the algorithms which, as we mentioned before, Newton worked out for problems of tangents, maxima and minima, and curvature. He even at one time introduced special notations in this connection (see *Papers*, vol. 1, 289–294), writing $\mathscr{X}$ for the left hand side of the equation of the curve (with the right hand side zero). He then wrote $\cdot\mathscr{X}$ and $\mathscr{X}\cdot$ for what we would write as xf_x and yf_y respectively (the so-called 'homogeneous partial derivatives'), using further symbols for homogeneous higher-order partial derivatives occurring in connection with curvature. However, the connection of Newton's $\cdot\mathscr{X}$ and $\mathscr{X}\cdot$ with modern partial derivatives should not be considered without some qualifications; he defined them formally as modifications of the formula $\mathscr{X}$, and he did not explicitly view $\mathscr{X}$ as a function of two variables which assumes also other values than the zero in the equation.

With these algorithms, and further finesses which we cannot go into here, Newton was able to solve what he formulated as one of the two fundamental problems in infinitesimal calculus: given the fluents and their relations, to find the fluxions.

The second problem is the converse of the first: given the relation of the fluxions, to find the relation of the fluents. Transposed in modern terminology, this means: given a differential equation, to find its solution. This of course is a much harder problem than the first. Newton did more about the problem than formulate it; his integral tables, already mentioned, form a means toward its solution, and he also studied various individual differential equations (or rather, fluxional equations).

As we have seen in the previous section, Newton's calculus was not to have the influence which Leibniz's achieved. Therefore, within the space and organisation of this chapter, we must leave it at this short summary of the fluxional calculus and some more remarks on its foundations in section 2.10, turning now our attention to the more successful rival, the Leibnizian calculus.

2.3. *The principal ideas in Leibniz's discovery*

One of the most precious documents of the Leibniz archive at Hanover is a set of mathematical manuscripts dated 25, 26 and 29 October, and 1 and 11 November, 1675.[1] On these sheets Leibniz wrote down his thoughts, more or less as they came to him, during a study of that most important problem of 17th-century mathematics: to find methods for the quadrature of curves. In the course of these studies he came to introduce the symbols '$\int$' and 'd', to explore the operational rules which they obey in formulas, and to apply them in translating many geometrical arguments about the quadrature of curves into symbols and formulas. In short, these manuscripts contain the record of Leibniz's 'invention' of the calculus. We will discuss them in more detail below, but first we will mention three principal ideas which guided him in those fateful studies in 1675.

The first principal idea was a philosophical one, namely Leibniz's idea of a *characteristica generalis*, a general symbolic language, through which all processes of reason and argument could be written down in symbols and formulas; the symbols would obey certain rules of combination which would guarantee the correctness of the arguments. This idea guided him in much of his philosophical thinking; it also explains his great interest in notation and symbols in mathematics and in general his endeavour to translate mathematical statements and methods into formulas and algorithms. Thus, in studying the geometry of curves, he was interested in methods rather than in results, and especially in ways to transform these methods into algorithms performable with formulas. In short, he was looking for a *calculus* for infinitesimal-geometrical problems.

The second principal idea concerned difference sequences. In studying sequences $a_1, a_2, a_3, \ldots$, and the pertaining difference sequences $b_1 = a_1 - a_2$, $b_2 = a_2 - a_3$, $b_3 = a_3 - a_4, \ldots$, Leibniz had noted that

$$b_1 + b_2 + \ldots + b_n = a_1 - a_{n+1}. \tag{2.3.1}$$

This means that difference sequences are easily summed, an insight which he put to good use in solving a problem which Huygens suggested

[1] They are discussed in Hofmann *1949a*, and an English translation is given in Child *1920a*.

to him in 1672: to sum the series $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \ldots$, the denominators being the so-called 'triangular numbers' $r(r+1)/2$. He found that the terms can be written as differences,

$$\frac{2}{r(r+1)} = \frac{2}{r} - \frac{2}{r+1}, \tag{2.3.2}$$

and hence

$$\sum_{r=1}^{n} \frac{2}{r(r+1)} = 2 - \frac{2}{n+1}. \tag{2.3.3}$$

In particular, the series, when summed to infinity has sum 2. This result motivated him to study a whole scheme of related sum and difference sequences, which he put together in his so-called 'harmonic triangle' (figure 2.3.1), in which the oblique rows are successive difference sequences, so that their sums can be easily read off from the scheme (Leibniz *Writings*, vol. 5, 405: compare Hofmann *1949a*, 12; *1974a*, 20).

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
 & & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & & \\
 & & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & & \\
 & & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & & \\
 & \frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} & \\
\frac{1}{7} & & \frac{1}{42} & & \frac{1}{105} & & \frac{1}{140} & & \frac{1}{105} & & \frac{1}{42} & & \frac{1}{7} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Figure 2.3.1.

Leibniz's 'harmonic triangle'. The numbers in the n-th row are

$$\left[(n-1)\binom{n}{k}\right]^{-1}.$$

Summations can be read off from the scheme as, for example:

$$\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots = \frac{1}{2}.$$

These results were not exactly new, but they did make Leibniz aware that the forming of difference sequences and of sum sequences are mutually inverse operations. This principal idea became more significant when he transposed it to geometry. The curve in figure 2.3.2 defines a sequence of equidistant ordinates y. If their distance is 1, the sum of the y's is an approximation of the quadrature of the curve, and the difference of two successive y's yields approximately the slope of the pertaining tangent. Moreover, the smaller the unit 1 is chosen, the better the approximation. Leibniz concluded that if the unit could

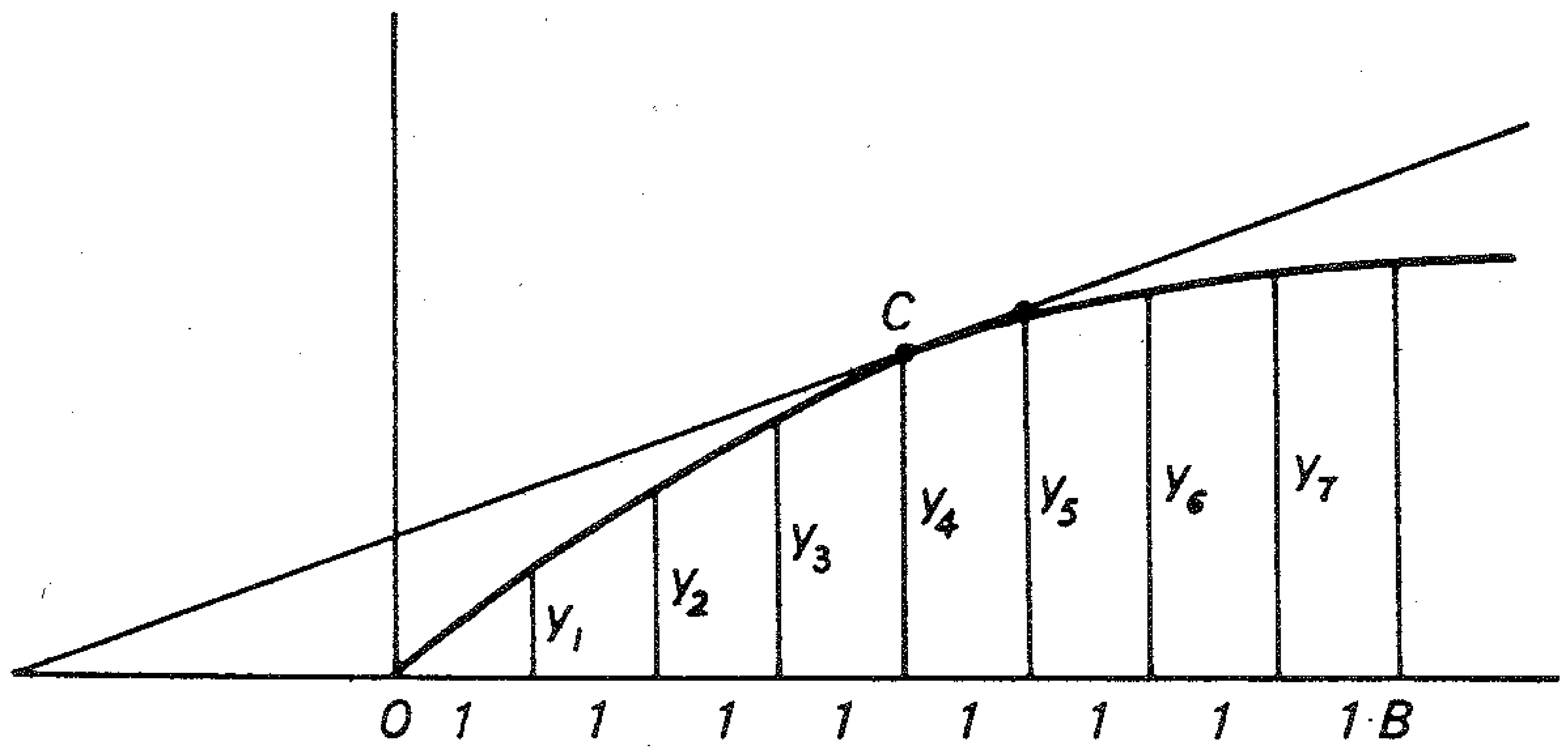

Figure 2.3.2.

be chosen *infinitely small*, the approximations would become exact: in that case the quadrature would be equal to the sum of the ordinates, and the slope of the tangent would be equal to the difference of the ordinates. In this way, he concluded from the reciprocity of summing and taking differences that the determination of quadratures and tangents are also mutually inverse operations.

Thus Leibniz's second principal idea, however vague as it was in about 1673, suggested already an infinitesimal calculus of sums and differences of ordinates by which quadratures and tangents could be determined, and in which these determinations would occur as inverse processes. The idea also made plausible that, just as in sequences the determination of differences is always possible but the determination of sums is not, so in the case of curves the tangents are always easily to be found, but not so the quadratures.

The third principal idea was the use of the 'characteristic triangle' in transformations of quadratures. In studying the work of Pascal, Leibniz noted the importance of the small triangle $cc'd$ along the curve in figure 2.3.3, for it was (approximately) similar to the triangles formed

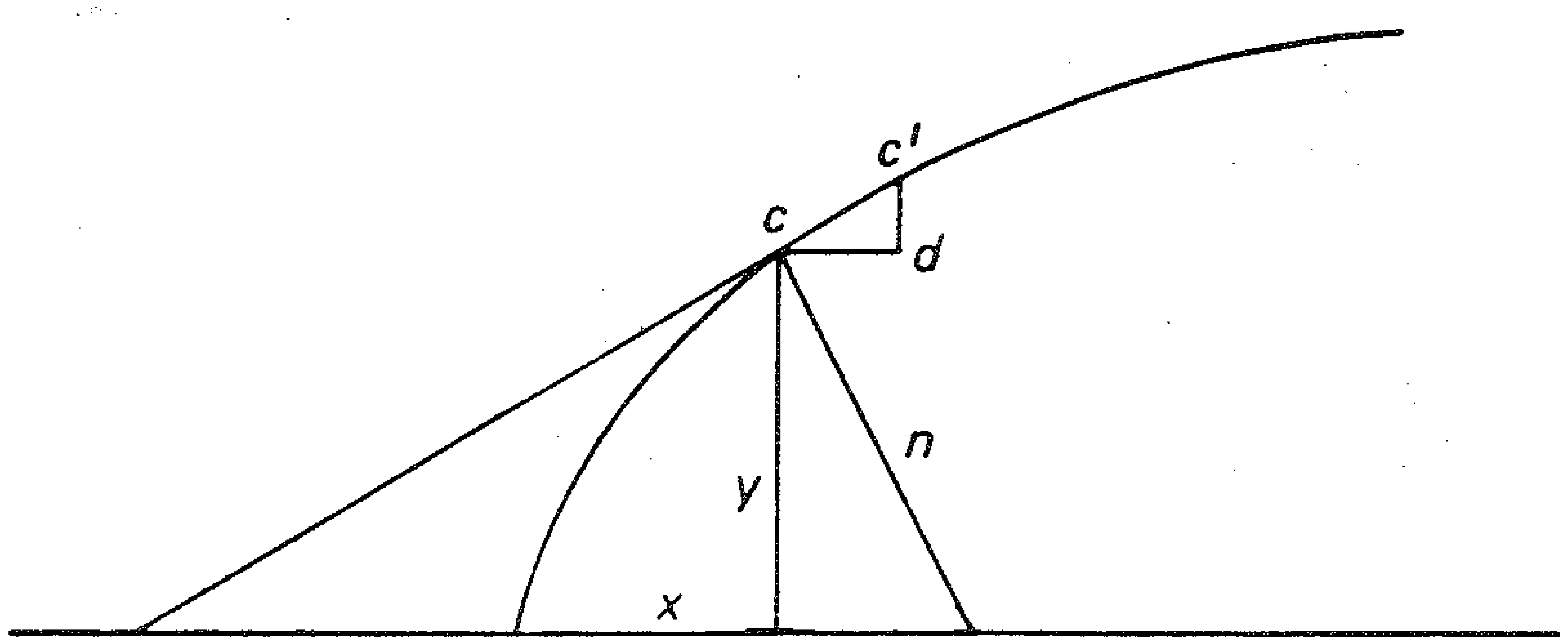

Figure 2.3.3.

by ordinate, tangent and sub-tangent, or ordinate, normal and sub-normal. The configuration occurs in many 17th-century mathematical works; Pascal's use of it concerned the circle. Leibniz saw its general use in finding relations between quadratures of curves and other quantities like moments and centres of gravity. For instance, the similarity of the triangles yields $cc' \times y = cd \times n$; hence

$$\sum cc' \times y = \sum cd \times n. \qquad (2.3.4)$$

The left hand side can be interpreted as the total moment of the curve arc with respect to the x-axis (the moment of a particle with respect to an axis is its weight multiplied by its vertical distance to the axis), whereas the right hand side can be interpreted as the area formed by plotting the normals along the x-axis.

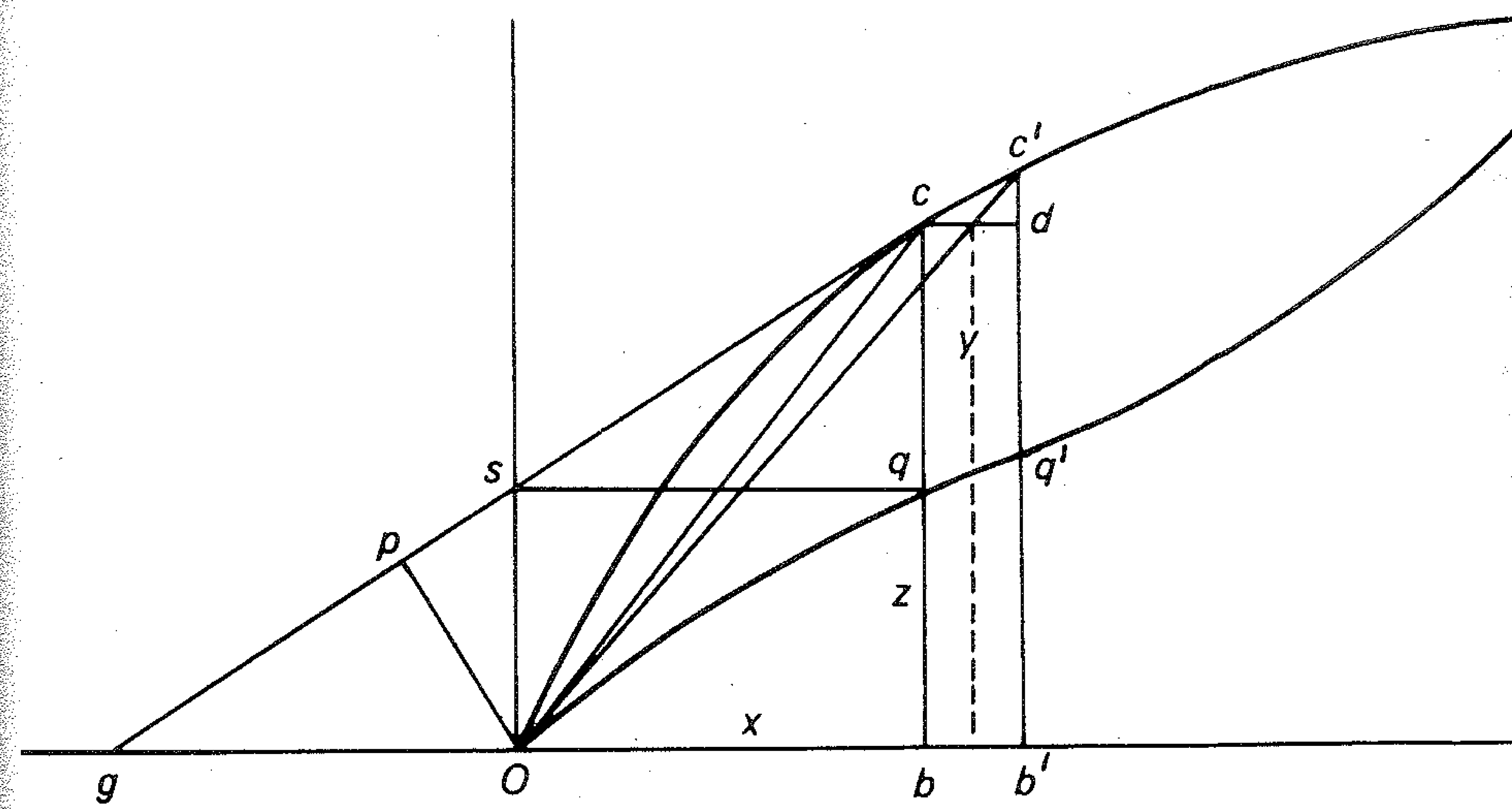

Figure 2.3.4.

As an example of Leibniz's use of the characteristic triangle, here is his derivation of a special transformation of quadratures which he called 'the transmutation' and which, for good reasons, he valued highly (compare Hofmann *1949a*, 32–35 (*1974a*, 54–60), and Leibniz *Writings*, vol. 5, 401–402). In figure 2.3.4 let the curve $Occ'C$ be given, with characteristic triangle cdc' at c. Its quadrature $\mathcal{Q} = \widehat{OCB}$, the sum of the strips $bcc'b'$, can also be considered as the sum of the triangles Occ' supplemented by the triangle OBC:

$$\mathcal{Q} = \sum \Delta Occ' + \Delta OBC. \qquad (2.3.5)$$

Now

$$\Delta Occ' = \tfrac{1}{2}cc' \times Op$$
$$= \tfrac{1}{2}cd \times Os$$

(since the characteristic triangle cdc' is similar to ΔOsp)

$$= \tfrac{1}{2}bqq'b'. \qquad (2.3.6)$$

Now for each c on $Occ'C$ we can find the corresponding q by drawing the tangent, determining s and taking $bq = Os$. Thus we form a new curve $Oqq'Q$, and we have from (2.3.5):

$$\mathcal{Q} = \tfrac{1}{2}\,(\text{quadrature } Oqq'Q) + \Delta OCB. \qquad (2.3.7)$$

This is Leibniz's transmutation rule which, through the use of the characteristic triangle, yields a transformation of the quadrature of a curve into the quadrature of another curve, related to the original curve through a process of taking tangents. It can be used in those cases where the quadrature of the new curve is already known, or bears a known relation to the original quadrature. Leibniz found this for instance to be the case with the general parabolas and hyperbolas (see section 1.3), for which the rule gives the quadratures very easily. He also applied his transmutation rule to the quadrature of the circle, in which investigation he found his famous arithmetical series for π:

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \ldots . \qquad (2.3.8)$$

The success of the transmutation rule also convinced him that the analytical calculus for problems of quadratures which he was looking for would have to cover transformations such as this one by appropriate symbols and rules.

The transmutation rule as Leibniz discovered it in 1673 belongs to the style of geometrical treatment of problems of quadrature which was common in the second half of the 17th century. Similar rules and methods can be found in the works of Huygens, Barrow, Gregory and others. Barrow's *Lectiones geometricae* (*1670a*), for instance, contain a great number of transformation rules for quadratures which, if translated from his purely geometrical presentation into the symbolism and notation of the calculus, appear as various standard alogrithms of the differential and integral calculus. This has even been used (by J. M. Child in his *1920a*) as an argument to give to Barrow, rather than Newton or Leibniz, the title of inventor of the calculus. However, this view can be sustained only when one disregards completely the effect of the translation of Barrow's geometrical text into analytical formulas. It is the very possibility of the analytical expression of methods, and hence

the understanding of their logical coherence and generality, which was the great advantage of Newton's and Leibniz's discoveries.

It is appropriate to illustrate this advantage by an example. To do this, I shall give a translation, with comments, of Leibniz's transmutation rule into analytical formulas.

The ordinate z of the curve $Oqq'Q$ is, by construction,

$$z = y - x\frac{dy}{dx} \qquad (2.3.9)$$

(note the use of the characteristic triangle). The transmutation rule states that, for $OB = x_0$,

$$\int_0^{x_0} y\,dx = \tfrac{1}{2}\int_0^{x_0} z\,dx + \tfrac{1}{2}x_0 y_0. \qquad (2.3.10)$$

Inserting z from (2.3.9), we find

$$\int_0^{x_0} y\,dx = \tfrac{1}{2}\int_0^{x_0}\left(y - x\frac{dy}{dx}\right)dx + \tfrac{1}{2}x_0 y_0$$
$$= \tfrac{1}{2}\int_0^{x_0} y\,dx - \tfrac{1}{2}\int_0^{x_0} x\frac{dy}{dx}\,dx + \tfrac{1}{2}x_0 y_0.$$

Hence

$$\int_0^{x_0} y\,dx + \int_0^{x_0} x\frac{dy}{dx}\,dx = x_0 y_0, \qquad (2.3.11)$$

so that we recognise the rule as an instance of 'integration by parts'.

Apart from the indication of the limits of integration $(0, x_0)$ along the $\int$-sign, the symbolism used above was found by Leibniz in 1675. The advantages of that symbolism over the geometrical deduction and statement of the rule are evident: the geometrical construction of the curve $Oqq'Q$ is described by a simple formula (2.3.9), and the formalism carries the proof of the rule with it, as it were. (2.3.11) follows immediately from the rule

$$d(x\,y) = x\,dy + y\,dx. \qquad (2.3.12)$$

These advantages, manipulative ease and transparency through the rules of the symbolism, formed the main factors in the success of Leibniz's method over its geometrical predecessors.

But we have anticipated in our story. So we return to October 1675, when the transmutation rule was already found but not yet the new symbolism.

2.4. *Leibniz's creation of the calculus*

In the manuscripts of 25 October–11 November 1675 we have a close record of studies of Leibniz on the problem of quadratures. We find him attacking the problem from several angles, one of these being the use of the Cavalierian symbolism ' omn.' in finding, analytically (that is, by manipulation of formulas) all sorts of relations between quadratures. ' Omn. ' is the abbreviation of ' omnes lineae ', ' all lines '; in section 1.10 it was represented by the symbol ' $\mathcal{O}$ '.

A characteristic example of Leibniz's investigations here is the following. In a diagram such as figure 2.4.1 he conceived a sequence of

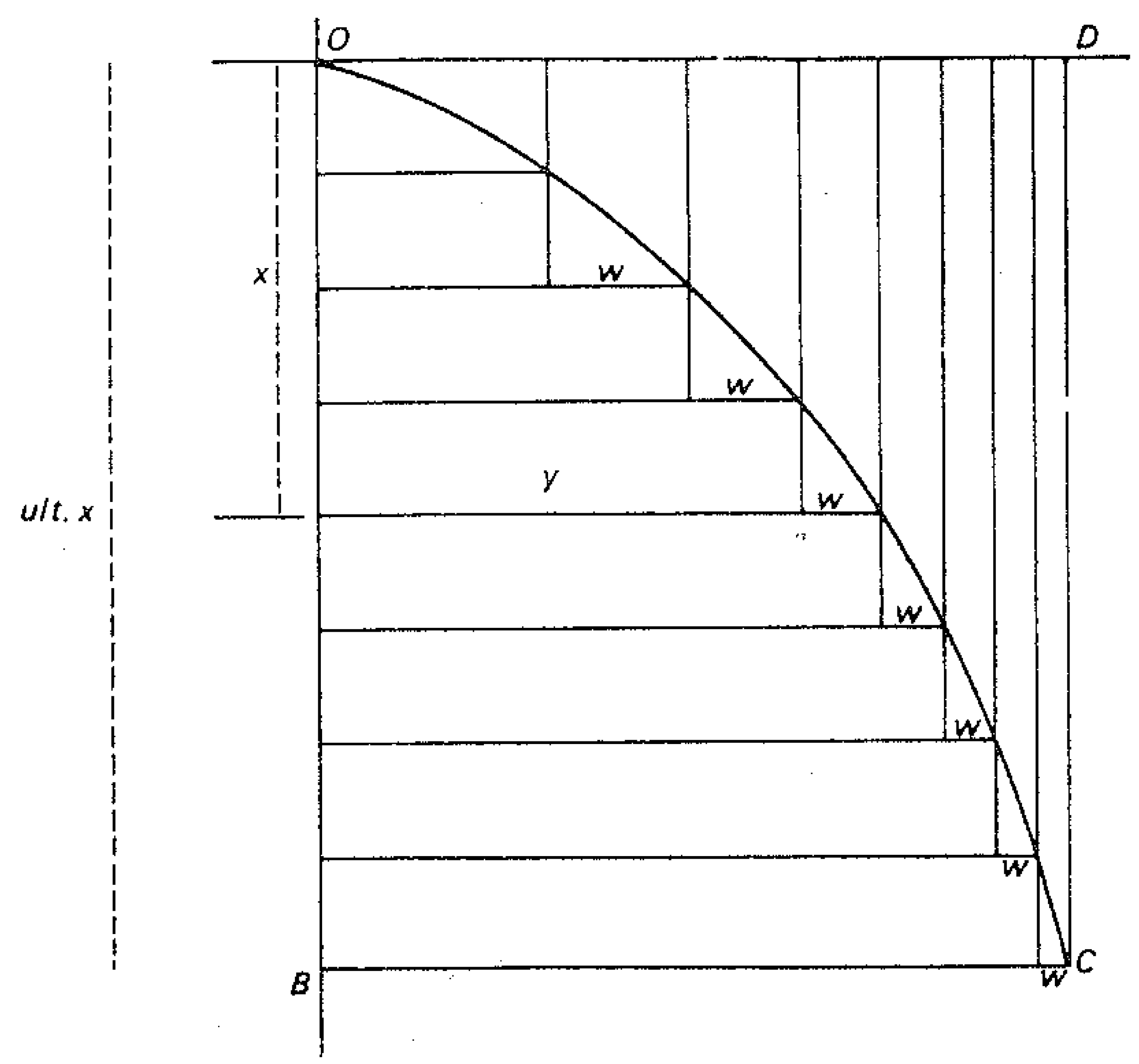

Figure 2.4.1.

ordinates y of the curve $\widehat{OC}$; the distance between successive ordinates is the (infinitely small) unit. The differences of the successive ordinates are called w. OBC is then equal to the sum of the ordinates y. The rectangles like $w \times x$ are interpreted as the moments of the differences w with respect to the axis OD (moment = weight × distance to axis). Hence the area OCD represents the total moment of the differences w. $\widehat{OCB}$ is the complement of $\widehat{OCD}$ within the rectangle $ODCB$, so that Leibniz finds that ' The moments of the differences about a straight line perpendicular to the axis are equal to the complement of the sum of the terms ' (Child *1920a*, 20). The ' terms ' are the y. Now w is the difference sequence of the sequence of ordinates y; hence, conversely, y is the sum-sequence of the w's, so that we may eliminate y

and consider only the sequence w and its sum-sequences, which yields: ' and the moments of the terms are equal to the complement of the sum of the sums ' (*ibid.*). Here the ' terms ' are the w. Leibniz writes this result in a formula using the symbol ' Omn. ' for what he calls ' a sum '. We give the formula as he gave it, and we add an explanation under the accolades; $\sqcap$ is his symbol for equality, ' ult. x ' stands for *ultimus* x, the last of the x, that is, OB, and he uses overlining and commas where we would use brackets (*ibid.*):

$$\underbrace{\text{omn. } \overline{xw}}_{\substack{\text{moments of}\\ \text{the terms } w}} \sqcap \underbrace{\underbrace{\text{ult. } x, \overline{\text{omn. } w},}_{\text{total}} - \underbrace{\overline{\text{omn. omn. } w}}_{\substack{\text{sum of the sums}\\ \text{of the terms}}}}_{\substack{\text{complement of the sum of the sums}\\ \text{of the terms}}} \tag{2.4.1}$$

(Compare the form of (2.4.1) with that of (2.3.11).) Immediately he sees the possibility to obtain from this formula, by various substitutions, other relations between quadratures. For instance, substitution of $xw = a$, $w = a/x$ yields

$$\text{omn. } a \sqcap \text{ult. } x, \text{ omn. } \frac{a}{x} - \text{omn. omn. } \frac{a}{x}, \tag{2.4.2}$$

which he interprets as an expression of the ' sum of the logarithms in terms of the quadrature of the hyperbola ' (*ibid*,. 71). Indeed, omn. a/x is the quadrature of the hyperbola $y = a/x$, and this quadrature is a logarithm, so that omn. omn. a/x is the sum of the logarithms.

We see in these studies an endeavour to deal analytically with problems of quadrature through appropriate symbols and notations, as well as a clear recognition and use of the reciprocity relation between difference and sum sequences. In a manuscript of some days later, these insights are pushed to a further consequence. Leibniz starts here from the formula (2.4.1), now written as

$$\text{omn. } xl \sqcap x \text{ omn. } l - \text{omn. omn. } l. \tag{2.4.3}$$

He stresses the conception of the sequence of ordinates with infinitely small distance: ' . . . l is taken to be a term of the progression, and x is the number which expresses the position or order of the l corresponding to it; or x is the ordinal number and l is the ordered thing ' (*ibid.*, 80). He now notes a rule concerning the dimensions in formulas like (2.4.3), namely that omn., prefixed to a line, such as l, yields an area (the quadrature); omn., prefixed to an area, like xl, yields a solid, and so on. Such a law of dimensional homogeneity was well-known from the Cartesian analysis of curves, in which the formulas must consist of terms all of the same dimension. (In (2.4.3) all terms are of three dimensions, in $x^2+y^2=a^2$ all terms are of two dimensions; an expression like a^2+a is, if dimensionally interpreted, unacceptable, for it would express the sum of an area and a line.)

This consideration of dimensional homogeneity seems to have suggested to Leibniz to use a single letter instead of the symbol ' omn. ', for he goes on to write: ' It will be useful to write $\int$ for omn, so that $\int l$ stands for omn. l or the sum of all l's ' (*ibid.*). Thus the $\int$-sign is introduced. ' $\int$ ' is one of the forms of the letter ' s ' as used in script (or italics print) in Leibniz's time: it is the first letter of the word *summa*, sum. He immediately writes (2.4.3) in the new formalism:

$$\int xl = x\int l - \int\int l; \tag{2.4.4}$$

he notes that

$$\int x = x^2/2 \quad \text{and} \quad \int x^2 = x^3/3, \tag{2.4.5}$$

and he stresses that these rules apply for ' series in which the differences of the terms bear to the terms themselves a ratio that is less than any assigned quantity ' (*ibid.*), that is, series whose differences are infinitely small.

Some lines further on we also find the introduction of the symbol ' d ' for differentiating. It occurs in a brilliant argument which may be rendered as follows: The problem of quadratures is a problem of summing sequences, for which we have introduced the symbol ' $\int$ ' and for which we want to elaborate a *calculus*, a set of useful algorithms. Now summing sequences, that is, finding a general expression for $\int y$ for given y, is usually not possible, but it *is* always possible to find an expression for the differences of a given sequence. This finding of differences is the reciprocal calculus of the calculus of sums, and therefore we may hope to acquire insight in the calculus of sums by working out the reciprocal calculus of differences. To quote Leibniz's own words (*ibid.*, 82):

> Given l, and its relation to x, to find $\int l$. This is to be obtained from the contrary calculus, that is to say, suppose that $\int l = ya$. Let $l = ya/d$; then just as $\int$ will increase, so d will diminish the dimensions. But $\int$ means a sum, and d a difference. From the given y, we can always find y/d or l, that is, the difference of the y's.

Thus the ' d '-symbol (or rather the symbol ' $1/d$ ') is introduced Because Leibniz interprets $\int$ dimensionally, he has to write the ' d ' in the denominator: l is a line, $\int l$ is an area, say ya (note the role of ' a ' to make it an area), the differences must again be lines, so we must write ' ya/d '. In fact he soon becomes aware that this is a notational disadvantage which is not outweighed by the advantage of dimensional

interpretability of $\int$ and d, so he soon writes ' $d(ya)$ ' instead of ' ya/d ' and henceforth re-interprets ' d ' and ' $\int$ ' as dimensionless symbols. Nevertheless, the consideration of dimension did guide the decisive steps of choosing the new symbolism.

In the remainder of the manuscript Leibniz explores his new symbolism, translates old results into it and investigates the operational rules for $\int$ and d. In these investigations he keeps for some time to the idea that $d(uv)$ must be equal to $du\,dv$, but finally he finds the correct rule

$$d(uv) = u\,dv + v\,du. \tag{2.4.6}$$

Another problem is that he still for a long time writes $\int x$, $\int x^2$, . . . for what he is later to write consistently as $\int x\,dx$, $\int x^2\,dx$, ...

A lot of this straightening out of the calculus was still to be done after 11 November 1675; it took Leibniz roughly two years to complete it. Nevertheless, the manuscripts which we discussed contain the essential features of the new, the Leibnizian, calculus: the concepts of the differential and the sum, the symbols d and $\int$, their inverse relation and most of the rules for their use in formulas.

Let us summarise shortly the main features of these Leibnizian concepts (compare Bos *1974a*, 12–35). The *differential* of a variable y is the infinitely small difference of two successive values of y. That is, Leibniz conceives corresponding sequences of variables such as y and x in figure 2.4.2. The successive terms of these sequences lie infinitely close. dy is the infinitely small difference of two successive ordinates y, dx is the infinitely small difference of two successive abscissae x, which, in this case, is equal to the infinitely small distance of two successive y's. A sum (later termed ' integral ' by the Bernoullis) like $\int y\,dx$ is the sum of the infinitely small rectangles $y \times dx$. Hence the quadrature of the curve is equal to $\int y\,dx$.

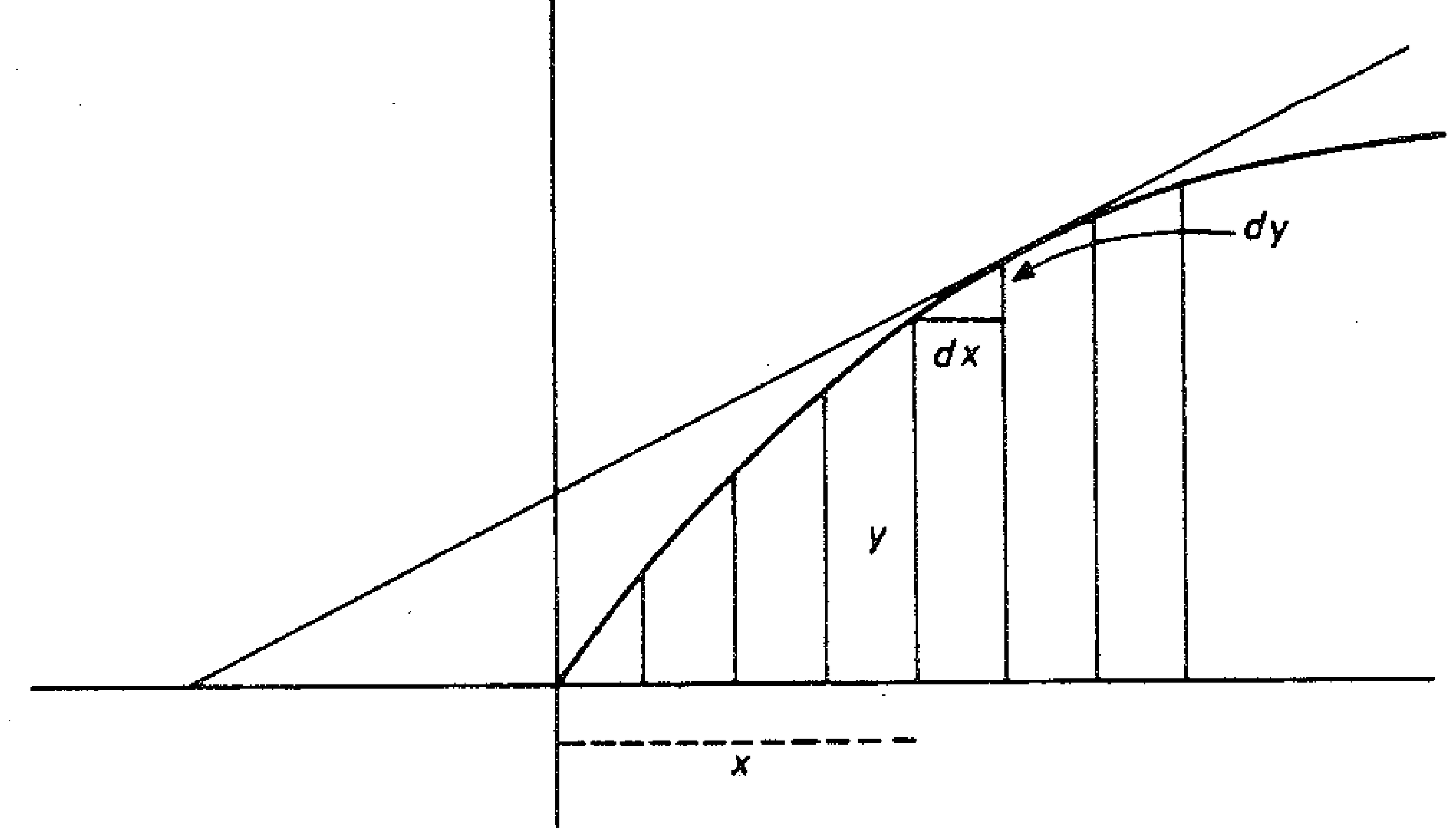

Figure 2.4.2.

Leibniz was rather reluctant to present his new calculus to the general mathematical public. When he eventually decided to do so, he faced the problem that his calculus involved infinitely small quantities, which were not rigorously defined and hence not quite acceptable in mathematics. He therefore made the radical but rather unfortunate decision to present a quite different concept of the differential which was not infinitely small but which satisfied the same rules. Thus in his first publication of the calculus, the article 'A new method for

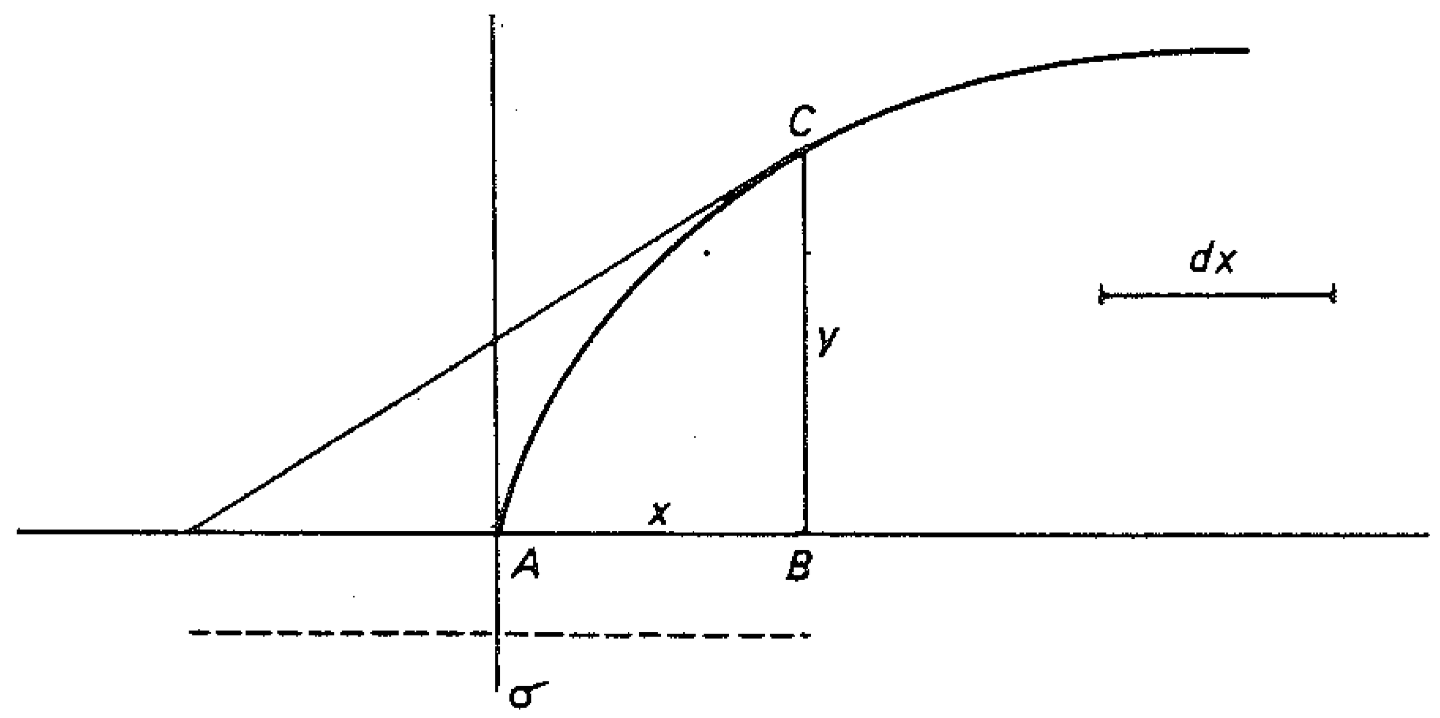

Figure 2.4.3.

maxima and minima as well as tangents' (*1684a*) in the issue for October 1684 of the *Acta*, he introduced a fixed finite line-segment (see figure 2.4.3) called dx, and he defined the dy at C as the line-segment satisfying the proportionality

$$y : \sigma = dy : dx, \qquad (2.4.7)$$

σ being the length of the sub-tangent, or

$$dy = \frac{y}{\sigma}\, dx. \qquad (2.4.8)$$

So defined, dy is also a finite line-segment. Leibniz presented the rules of the calculus for these differentials, and indicated some applications. In an article published two years later (*1686a*) he gave some indications about the meaning and use of the ∫-symbol. This way of publication of his new methods was not very favourable for a quick and fruitful reception in the mathematical community. Nevertheless, the calculus was accepted, as we shall see in the following sections.

2.5. *l'Hôpital's textbook version of the differential calculus*

Leibniz's publications did not offer an easy access to the art of his new calculus, and neither did the early articles of the Bernoullis. Still, a

good introduction appeared surprisingly quickly, at least to the differential calculus, namely l'Hôpital's *Analyse* (*1696a*).

As a good textbook should, the *Analyse* starts with definitions, o variables and their differentials, and with postulates about these differentials. The definition of a differential is as follows: 'The infinitely small part whereby a variable quantity is continually increased or decreased, is called the differential of that quantity' (ch. 1). Fo further explanation l'Hôpital refers to a diagram (figure 2.5.1), in which

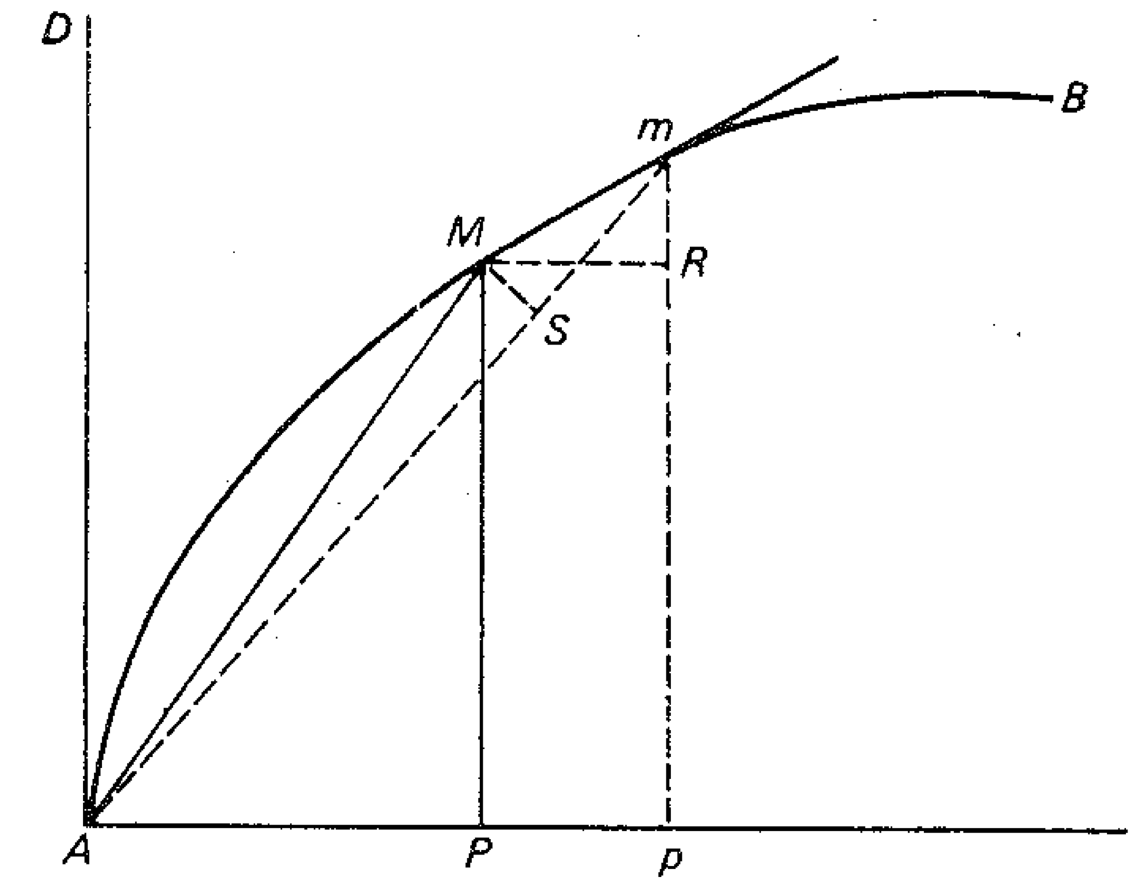

Figure 2.5.1.

with respect to a curve AMB, the following variables are indicated abscissa $AP = x$, ordinate $PM = y$, chord $AM = z$, arc $\widehat{AM} = s$ and quadrature $\widehat{AMP} = \mathcal{Q}$. A second ordinate pm 'infinitely close' to PM is drawn, and the differentials of the variables are seen to be: $dx = Pp$ $dy = mR$, $dz = Sm$, $ds = Mm$ (the chord Mm and the arc Mm are taken to coincide) and $d\mathcal{Q} = MPpm$. l'Hôpital explains that the 'd' is a special symbol, used only to denote the differential of the variable written after it. The small lines Pp, mR, . . . in the figure have to be considered as 'infinitely small'. He does not enter into the question whether such quantities exist, but he specifies, in the two postulates how they behave (*ibid.*):

> Postulate I. Grant that two quantities, whose difference is an infinitely small quantity, may be used indifferently for each other: or (which is the same thing) that a quantity, which is increased or decreased only by an infinitely smaller quantity, may be considered as remaining the same.

This means that AP may be considered equal to Ap (or $x = x + dx$) MP equal to mp ($y = y + dy$), and so on.

The second postulate claims that a curve may be considered as the assemblage of an infinite number of infinitely small straight lines, or equivalently as a polygon with an infinite number of sides. The first postulate enables l'Hôpital to derive the rules of the calculus, for instance:

$$\left.\begin{aligned} d(xy) &= (x+dx)(y+dy)-xy \\ &= x\,dy + y\,dx + dx\,dy \\ &= x\,dy + y\,dx \end{aligned}\right\} \quad (2.5.1)$$

'because $dx\,dy$ is a quantity infinitely small, in respect of the other terms $y\,dx$ and $x\,dy$: for if, for example, you divide $y\,dx$ and $dx\,dy$ by dx, we shall have the quotients y and dy, the latter of which is infinitely less than the former' (*ibid.*, ch. 1, para. 5). l'Hôpital's concept of differential differs somewhat from Leibniz's. Leibniz's differentials are infinitely small differences between successive values of a variable. l'Hôpital does not conceive variables as ranging over a sequence of infinitely close values, but rather as continually increasing or decreasing; the differentials are the infinitely small parts by which they are increased or decreased.

In the further chapters l'Hôpital explains various uses of differentiation in the geometry of curves: determination of tangents, extreme values and radii of curvature, the study of caustics, envelopes and various kinds of singularities in curves. For the determination of tangents he remarks that postulate 2 implies that the infinitesimal part Mm of the curve in figure 2.5.2, when prolonged, gives the tangent.

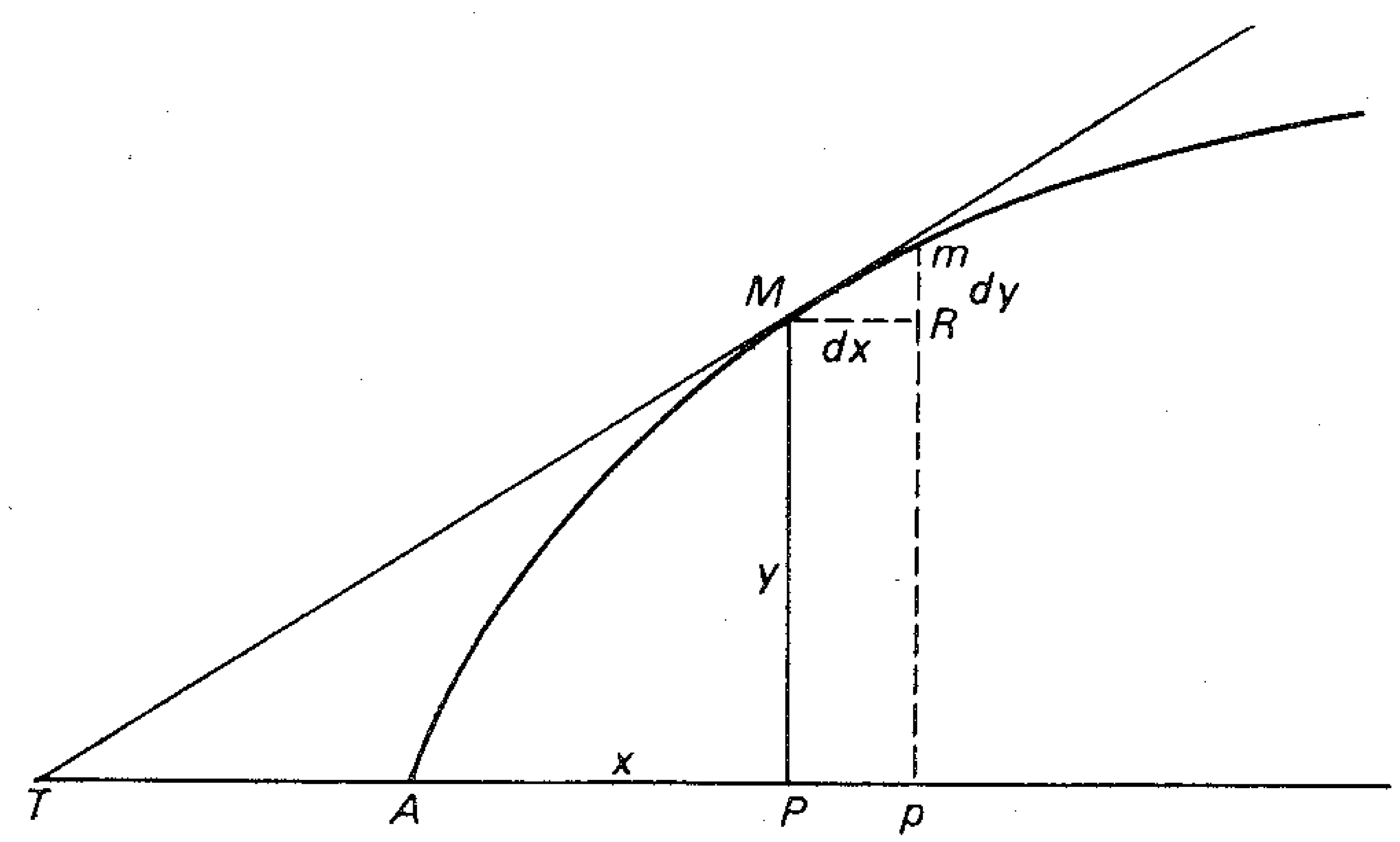

Figure 2.5.2.

Therefore $Rm : RM$, or $dy : dx$, is equal to $y : PT$, so that $PT = y(dx/dy)$, and the tangent can be constructed once we have determined $y\,dx/dy$ (*ibid.*, ch. 2, para. 9):

Now by means of the difference of the given equation you can obtain a value of dx in terms which all contain dy, and if you multiply by y and divide by dy you will obtain an expression for the sub-tangent PT entirely in terms of known quantities and free from differences, which will enable you to draw the required tangent MT.

To explain this, consider for example the curve $ay^2 = x^3$. The 'difference of the equation' is derived by taking differentials left and right:

$$2ay\,dy = 3x^2\,dx. \quad (2.5.2)$$

dx can now be expressed in terms of dy:

$$dx = \frac{2ay}{3x^2}\,dy. \quad (2.5.3)$$

Hence

$$PT = \frac{y\,dx}{dy} = y\,\frac{2ay}{3x^2} = \frac{2ay^2}{3x^2}, \quad (2.5.4)$$

which provides the construction of the tangent.

The 'difference of the equation' is a true differential equation, namely an equation between differentials. l'Hôpital considers expressions like 'dy/dx' actually as quotients of differentials, not as single symbols for derivatives.

2.6. *Johann Bernoulli's lectures on integration*

In 1742, more than fifty years after they were written down, Johann Bernoulli published his lectures to l'Hôpital on 'the method of integrals' in his collected works (Bernoulli *1691a*), stating in a footnote that he omitted his lectures on differential calculus as their contents were now accessible to everyone in l'Hôpital's *Analyse*. His lectures may be considered as a good summary of the views on integrals and their use in solving problems which were current around 1700.

Bernoulli starts with defining the integral as the inverse of the differential: the integrals of differentials are those quantities from which these differentials originate by differentiation. This conception of the integral—the term, in fact, was introduced by the Bernoulli brothers—differs from Leibniz's, who considered it as a sum of infinitely small quantities. Thus, in Leibniz's view, $\int y\,dx = \mathcal{Q}$ means that the sum of the infinitely small rectangles $y \times dx$ equals $\mathcal{Q}$; for Bernoulli it means that $d\mathcal{Q} = y\,dx$.

Bernoulli states that the integral of $ax^p\,dx$ is $(a/(p+1))x^{p+1}$, and he