The Modern Analysis of the Infinite: Introductory Notes

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PART I. BACKGROUND IDEAS

1. Introduction

The purpose of these notes is to set out simply and clearly the core technical ideas in the modern mathematical theory of the infinite. The goal is not to train readers in the tools of mathematics or logic. Here there are few proofs or technical exercises, and those that occur are offered only because they help explain ideas. Rather the objective is to provide a reference tool for the general reader, above all philosophy students and those interested in ideas. These notes therefore are mainly definitions and discussions of definition. Becuse it is the nature of the subject, the definitions are progressive, later concepts being defined in terms of earlier ones. To some degree it is possible to pick up the material and start reading in the middle. But to do so it will be necessary to master the rudiments of set theoretic notation that is used throughout. The basics of set theory and other background ideas are set out in Part I. Part II develops the properties of the infinite as exemplified in the various number systems invented by mathematicians in part to explain just these ideas: the natural numbers, integers, rationals, reals, hyperreals and metric spaces. It is possible to read their definitions independently so long as it is done with the understand that the definitions have hostages, concepts that themselves need definitions. Among these especially are the relations like ≤ and operations like addition and multiplication that are defined for one type in terms of those on the previous type. Part III sets out the elementary ideas of the theory of transfinite sets, both ordinals and cardinals.

In the nineteenth and twentieth centuries mathematicians discovered that it was possible to replace earlier philosophical accounts of the infinite with a precisely defined theory. This theory both provides for general accounts of what infinite sets are and general definitions of the various kinds of numerical structures that illustrate their properties. There is a deliberate progression both in the order of ideas defined and an order of the results the definitions yield. In particular is was discovered how to define progressively more powerful kinds of "numbers" which at each stage exhibit some new and important property of "the infinite." For example, the natural numbers (0,1,2,…) allow us to count and describe a set as infinite if in counting it we never come to an end. Unlike the natural numbers, the integers (positive and negative whole numbers, plus 0) allow us to count backwards as well as forward, in a way that exhibits more of the infinite structure of space and time. The rationals (ratios of integers) unlike the integers are "dense" like space and time – in between any two there is another, and this divisibility goes on forever. The real numbers (rationals plus irrational numbers, like $\sqrt{2}$ and π) exhibit even more of the structure of space and time than do the rationals because they are also "continuous" in the sense that they allow any line to be cut at any point such that its neighbors both above and below approach every closer and closer to it, forever. We shall even have occasion to mention a particularly abstract variety of number, the hyperreals, which have the property that some of them are literally infinitesimal, so infinitely small that they have no measurable size. Lastly, we shall review general definitions of the infinite that abstract from numbers entirely and depend only on the notion of order and counting. These are the so-called ordinal and cardinal numbers studied in set theory. We shall discover that among these there are in fact multiple levels of the infinite – some infinite sets are bigger than others. There are infinities of larger and larger infinities. Moreover, these the two notions – order and counting – lead to two different but related notions of the infinite.

This entire progression, in which one type of number is used to built up a yet more powerful type, all starts with the basic idea of set. We shall begin then with a short introduction to the ideas of elementary, also called "naïve," set theory.

2. Metaphysics, Universals and Sets

From a philosophical perspective the story of sets began long ago in ancient Greece. The first philosophers posed a famous problem that goes under several names, *the problem of the one and the many*, or *the problem of sameness and difference*, but is probably best know as *the problem of universals*. Briefly put, it seeks an answer to the question: how can two distinct things be the same and yet different. For example, how can Socrates and Plato, who are distinct people, both be human or both be white? In later philosophy the problem was rephrased as one about the truth of propositions: under what conditions are two subject-predicate sentences with the same subject term but different predicate terms both true together? For example, under what conditions are the propositions *Socrates is a human* and *Plato is a* human, or the propositions *Socrates is white* and *Plato is white* both be true together.

The traditional answer is to posit the existence of special "explanatory entities." The technique is part of scientific method. To explain puzzling phenomena philosophers and even scientists often posit entities attributing to them special laws for the purpose. For example, in philosophy creation has been explained by hypothesizing the existence of a god, and likewise life is "explained" by positing the soul. In science the phenotypic behavior of red and white sweet peas (the ratio of the populations of different colors differ according to a set patter over generations) was explained by Mendel by positing the existence of genes governed by laws of dominance and recession. Pasture explained fermentation by positing an unseen catalytic agent. In atomic theory observed chemical reactions are explained by positing the table of elements with their atomic properties of combination.

The problem of sameness and difference too was "solved" by positing the existence of appropriate entities. First are posited ordinary things. These are the familiar objects that make up the world. They obey the "law" that they can be named by the subject terms (proper nouns) of propositions. In technical metaphysics such objects came to be called *substances*. In addition a second category of entity is posited. These came to be called *universals*. They obey the unusual "law' that they can be instantiated wholly and completely in more than one substance at a time. An explanation of why Socrates and Plato can both be human yet different, is then possible: there are universals like *humanity* and *whiteness* that inhere simultaneously in many substances. The propositions *Socrates is human* and *Plato is human* are both true together because the subject terms stand for different substances but the predicate terms stand for the same universal, humanity. Likewise in *Socrates is white* and *Plato is white* the predicate stands for the same universal, whiteness. Hence Socrates and Plato are the same. On the other hand, there are other universals like *snub-nosed* that inheres in some of these (Socrates) but not others (Plato). Hence *Socrates is snub-nosed* is true but *Plato is snub-nosed* is false. Hence Socrates and Plato are different.

But the status and real existence of universals was highly debated in philosophy. Those who believe in the existence of universals are called *realists* (they think universals really exist) and those who doubt their existence are called *nominalists* (there are "names" like the predicates *humanity* and *white* but there is not universal in the world to which they correspond). Realists face many difficulties. First though everybody admits the existence of normal objects (substances), universals are counter-intuitive. Nobody would have thought they existed if they had not be posited for the purpose of the explanation. Second, explanatory entities are accepted in science only as part of a larger theory within which they are defined and their properties explained. Moreover, in the empirical sciences at least the larger theory should be open to experimental confirmation or refutation. Nobody could explain, however, what universals are beyond the observation that they exist to solve the problem they were introduced to solve. The are *ad hoc* inventions about which there is no independent knowledge. Another problem is that at a basic level the theory seems incoherent. How can one and the same universal be wholly and completely "in" more than one substance at the same time? Ordinary objects cannot be in two places at once. How can universals? William of Ockham, a famous foe of universals, noted that if Socrates were to be annihilated so presumably would all the parts that make him up, including the universal humanity if it is wholly and completely "in" Socrates. God, Ockham observes, is omnipotent and could certainly annihilate Socrates and all his parts. God could meanwhile spare Plato. But if God does annihilates Socrates together with his parts, he annihilates the universal humanity, which is a wholly and completely "in" Socrates. But the universal is in Plato as well. Hence a key "part" of Plato would cease to exist, and so too would presumably Plato himself. Hence annihilating Socrates seems to entail annihilating Plato. Hence it follows from the theory of universals that

God's power is limited and that he is not omnipotent. Ockham regarded this argument as a reduction to the absurd of the theory of universals.

Set theory may be regarded as a replacement for universals that avoids many of their problems. Rather than universals, set theory posits the existence of sets and their elements. Unlike the earlier account, set theory states "laws" that detail the properties of sets. These are its axioms, definitions, and the theorems they entail. The theory succeeds in meeting the original explanatory goal of universals because sameness and difference can be explained in terms of sets. By convention the Greek letter $\in$ (epsilon) is used to indicate set membership: $x \in A$ is read *x is a member A* or *x is an element of A.* (Epsilon is used because set membership is one of the senses of the verb *to be,* as in *Socrates is a human*, and in Greek the verb *to be* (enai) begins with $\in$). Socrates and Plato are the same because they are both members the same set. Indeed they are members of many sets. For example, they are both members of the set of humans and the set of white things. They are different because there are other sets that contain one but not the other. For example, Socrates but not Plato is in the set of snub-nosed things.

Sets can also be used to explain how the relevant propositions are true and false. Let the subject terms *Socrates* and *Plato* stand for the individual men, and let the predicate terms *human, white,* and *snub-nosed* stand for their respective sets of humans, white things and snub-nosed things. Hence in the actual world the following propositions are all true:

Socrates $\in$ *human,*
Plato $\in$ *human,*
Socrates $\in$ *white,*
Plato $\in$ *white,* and
Socrates $\in$ *snub-nosed*

However the following proposition is false:

Plato $\in$ *snub-nosed*

Set theory moreover is formulated in a rich theory. It has carefully stated axioms and definitions from which an entire body of results can be logically proven as theorems, many of which are interesting in their own right or useful for solving other problems in mathematics. In short, it meets the rigorous standards for a theory in mathematics of the sort science aspires to. Indeed set theory is used extensively in mathematics as the general foundational theory in which other important mathematical theories are developed and proved. One of the major applications of set theory is to the theory of numbers and the infinite. Accordingly, before taking up the topic of the infinite itself, we must introduce the relevant parts of set theory.

3. Axiom Systems

An axiom system consists of a set of basic propositions called ***axioms*** that are assumed to be true. In mathematics these are carefully formulate, often in a symbolic language. From the axiom other propositions called ***theorems*** are deduced one at a time by the rules of logic. A series of propositions each of which is an axiom or a theorem, or follows from previous propositions is the list by a rule of logic is called a ***proof***. Within an axiom system it is possible to set out clearly exactly in what sense an idea can be explained.

Ideas are "explained" in two ways. The first is in what is called an *explicit definition*. A definition of this sort is really a declaration that some group of symbols is to function as an abbreviation for a longer set. What they say is that whenever you see the abbreviation strictly speaking it should be erased and replaced by the longer expression it abbreviates. Propositions in which no abbreviations occur are said to be in ***primitive notation***. Abbreviations of this sort are generally laid out in one of two syntactic forms, as either a biconditionals or an identity.

A biconditionals is a formula in which two propositions are connected by the words ***if and only if***, which are themselves abbreviated in mathematics books by the expression ***iff***. In logic books *iff* is often expressed by the symbol $\leftrightarrow$, $\Leftrightarrow$, or $\equiv$. Syntactically the expressions that flank *iff* are propositions, i.e. entire declarative sentences that are either true or false. At the risk of getting ahead of ourselves it may be useful at this point to look as some real examples from logic of abbreviative definitions that make use of *iff.* In these definitions the subscript *def* is added to the *iff* to indicate that it is being used in a definition. The technical symbolism is translated into English. You should be looking at the way in which a shorter expression is used to abbreviate a longer one, rather than concerting yourself with what the concepts in question really mean. In reading these it may help to know that $\vee$ is the symbol of *or,* $\sim$ is the symbol for *not,* $\exists$ symbolizes *there exists a,* $\Diamond$ symbolizes *possibly*¸□ *symbolizes* necessarily, $\forall$ stands for *for all*, $\wedge$ represents *and*, and $\rightarrow$ stands for *if…then.*

$P\vee Q$ $iff_{\mathbf{def}}$ $\sim(\sim P\wedge\sim Q)$	*P or Q* means *not(not P and not Q*
x exists $iff_{\mathbf{def}}$ $\exists y(y=x)$	*x exists* means *there exists a y such that y=x.*
$\Diamond P$ $iff_{\mathbf{def}}$ $\sim\Box\sim P$	*possibly P* means *not necessarily not P*
$Q(1x \mid P(x))$ $iff_{\mathbf{def}}$ $\exists x((P(x)\wedge\forall y(P(y)\rightarrow y=x)\wedge Q(x))$	*the one and only P is Q* means *there exists an x such that it is P, and for any y if it is P it is the same as x, and x is Q*

Note that in all cases the abbreviation is shorter than what it abbreviates.

A second syntactic form of abbreviative definitions is identities. These are expressions of the form *s*=*t* where *s* and *t* are terms that stand for sets of elements of sets. Definitions of this sort are used extensively in set theory. Here are some examples that will be explained more fully shortly:

$-A =_{\text{def}} \{x \mid x\notin A\}$	*the complement of A* is *the set of all elements not in A*
$A\cup B =_{\text{def}} \{x \mid x\in A\vee x\in B\}$	*the union of A and B* is *the set of all objects that are either in A* or *in B*

Obviously, not all terms in a system can abbreviate others because the system has to start with the basic primitive vocabulary. Though the primitives cannot themselves be given abbreviative definitions, they can be explained in a second sense. In a sense they are explained by the axioms in which they occur. Indeed, the axioms are said to give the terms an ***implicit definition***.

4. The Axioms and Definitions of Naive Set Theory

The first axiom of set theory says in effect that any set that can be defined exists. The "definition" always consists of a sentence that describes what must be true of all and only the objects in the set. Syntactically the definition takes a special form. It always refers to an arbitrary object in the set by what is called a ***free variable***. This is usually a lower case letter from the end of the alphabet. Usually it is *x,y* or *z.* But it could be *u, v, w* or indeed any other letter if it is convenient to the discussion at hand. What is important is that the variable functions as a pronoun to pick out a potential member of the set, but that its specific referent is not yet fixed either by a prior antecedent in the syntax or by the context of its use. It is because its referent is not yet fixed that it is called ***free****.*

The sentence that makes use of the variable states what must be true of this object named by the variable if it is to be in the set. For example, the sentence *x is red* is an open sentence containing the free variable *x.* It says of *x* that it is red. Because this sentence makes use of a free variable is said to be ***open****.* Thus an ***open sentence*** is the technical term for a sentence that contains at least one free variable. We shall use sentences containing a single free variable to define sets, but note that in general a sentences may contain more than one free variable, for example, *x loves y*, or *x loves y but hates z.* Sentences with more than one free variable will be used later to define relations and functions.

Spotting free variables is easy if we are writing real sentences. We just look at its syntax to see if it contains any free variables. But in technical settings we sometimes want to abbreviate an open sentence and represent it by a single letter but at the same time we want to indicate that it contains some particular free variables. To do so, we adopt a couple of conventions. First, we use upper case letters like *P, Q and R* to stand for sentences. Next, to indicate that a sentence *P* is open and exactly which variables are free in it, we write *P* with those variables in parentheses immediate after it. For example, the notation $P(x)$ indicates that *P* is an open sentence that contains the free variable *x.* Likewise the notion $Q(x,y)$ indicates that *Q* is an open sentence that contains the two free variables *x* and *y.*

We can now make use of an open sentence to construct a name for a set. Such a name always has a special syntactic from. It has two parts, the open sentence and a prefix that indicates a set is being named. The prefix specifies the free variable used in its description. For example, the prefix *the set of all objects x such that* combines with the open sentence *x is red* to make up the set name *the set of all x such that x is red.* In technical settings the prefix *the set of all x such that …* is abbreviated $\{x \mid \ldots\}$. Thus $\{x \mid P(x)\}$ is read *the set of all x such that* $P(x)$. A set name of the form $\{x \mid P(x)\}$ is called a ***set abstract*** because it picks out the set by “abstracting” from its elements what they all have in common, namely the fact specified in the defining open sentence $P(x)$.

The first axiom of set theory can now be stated. It lays out in general terms the conditions under which a set exists: it exists if it a condition $P(x)$ for membership may be formulated in language. It says that if $P(x)$ is an open sentence, then there is an entity A (a set) such that x is an element of A if and only if $P(x)$ is true. The axiom can also be stated using set abstract notation: there exists a set $\{x \mid P(x)\}$ such that for any y, $y \in \{x \mid P(x)\}$ iff $P(y)$.

The second axiom of naïve set theory states the so-called “identity conditions” for sets, the conditions under which two sets are strictly identical. According to the axiom, two sets are identical if and only if they have the same members. Let us adopt the terminology that the set of objects that posses a property is called the properties ***extension***. Similarly let us call the set of all objects that make the open sentence $P(x)$ true is called the ***extension*** as well. Then another way of stating the axiom is that two sets are identical iff their defining properties or open sentences have the same extension.

Historically, it was controversial whether universals like *humanity* or *redness* “exist” because philosophers could not agree on a set of conditions spelling out the existence conditions of universals. Axiom 1 states flat out that sets exist under the condition that they are “definable”, i.e. they exist whenever there is an open sentence in the language that may be used to describe elements in the universe.

Traditional philosophers also had difficulty explaining when two properties were identical. Unlike sets properties need not be identical if they have the same extensions. It is perfectly possible for all and only red things to be square but that coincidence would not, according to philosophical usage at least, make the properties of redness and squareness the same. Axiom of 2 of set theory settles the matter for sets by declaring that two sets are identical if they have the same extensions.

The fact that set theory is set forth in terms of axioms and definitions makes it a genuine mathematical theory. Its axioms entail a large body of applicable to a wide variety of subjects. Set theory has in fact proved useful as a kind of general background theory in which large parts of modern mathematics is formulated. It is used in this way in these notes. The modern theory is formulated making use of its terms.

Axioms of Naïve Set Theory

1. **Principle of Abstraction.** Let $P(x)$ be an open sentence.
    - i. There is an set A such that for all x, $x \in A$ iff $P(x)$, (or equivalently)
    - ii. There is a set $\{x \mid P(x)\}$ such that for all y, $y \in \{x \mid P(x)\}$ iff $P(y)$

2. **Principle of Extensionality.** Let $P(x)$ and Let $Q(x)$ be open sentences.
    - i. $A=B$ iff for all x ($x \in A$ iff $x \in B$) (or equivalently)
    - ii. $\{x \mid P(x)\} = \{x \mid Q(x)\}$ iff for all y ($y \in \{x \mid P(x)\}$ iff $y \in \{x \mid Q(x)\}$)

Definitions			**Technical Name:**
$x \neq y$	*x is not identical to y*	$\sim(x=y)$	***non-identity*** or ***inequality***
$x \notin A$	*x is not an element of set A*	$\sim(x \in A)$	***non-membership***
$A \subseteq B$	*Everything in A is in B*	$\forall x(x \in A \rightarrow x \in B)$	*A* is a ***subset*** of *B*
$A \subset B$	*$A \subseteq B$ & some B is not in A*	$A \subseteq B \& \sim A=B$	*A* is a ***proper subset*** of *B*
$\varnothing$ or Λ	*set containing nothing*	$\{x \mid x \neq x\}$	the ***empty set***
V	*set containing everything*	$\{x \mid x=x\}$	the ***universal set***
$A \cap B$	*set of things in both A and B*	$\{x \mid x \in A \& x \in B\}$	the ***intersection*** of *A* and *B*
$A \cup B$	*set of things in either A or B*	$\{x \mid x \in A \vee x \in B\}$	the ***union*** of *A* and *B*
$A-B$	*set of things in A but not in B*	$\{x \mid x \in A \& x \notin B\}$	the ***relative complement*** of *B* in *A*
$-A$	*set of things not in A*	$\{x \mid x \notin A\}$	the ***complement*** of *A*
$\mathfrak{P}(A)$	*the set of subsets of A*	$\{B \mid B \subseteq A\}$	the ***power set*** of *A*

Here is an example of a "truth" of set theory and its proof.

Theorem. $A \cap B \subseteq B$.

Proof. Consider an arbitrary x and assume for conditional proof that $x \in A \cap B$. Hence by the definition of intersection, $x \in \{y \mid y \in A \wedge y \in B\}$. [Notice the need here to change variables to avoid confusion.] Hence by the Principle of Abstraction, $x \in A \wedge x \in B$. Hence by simplification, $x \in B$. Thus by conditional proof, it has be proven that $x \in A \cap B \rightarrow x \in B$. Since we have been general in x (*i.e.* since x has be "arbitrary"), we may universally generalize, $\forall x(x \in A \cap B \rightarrow x \in B)$. Thus, by the definition of subset, $A \cap B \subseteq B$. **QED.**

Below we list without proof a number of useful facts that follow directly from the definitions.

Theorems

1. $A \subseteq A$
2. $A \cap B \subseteq A \subseteq A \cup B$
3. $A=--A$
4. $\varnothing \subseteq A \subseteq V$
5. $\varnothing=-V$
6. $V=-\varnothing$
7. $(A \subseteq B \;\&\; B \subseteq A) \leftrightarrow A=B$
8. $A \cap B = B \cap A$
9. $A \cup B = B \cup A$
10. $A \cap (B \cap C) = (A \cap B) \cap C$
11. $A \cup (B \cup C) = (A \cup B) \cup C$
12. $A \cap A = A = A \cup A$
13. $-(A \cap B)=-A \cup -B$
14. $-(A \cup B)=-A \cap -B$
15. $-A=V-A$
16. $A \subseteq \mathfrak{P}(A)$

5. Relations and Functions

Just as *properties* were posited in traditional philosophy to name what two objects have in common, e.g. humanity or redness, philosophers also posited explanatory entities called ***relations*** to explain what two pairs of things have in common. The pairs Cain and Able, Castor and Pollex , Romulus and Remus all share the "relational" fact that they are brothers. Being a brother requires there be two people. They are said to stand in the brotherhood relation. Realists explain what Cain and Able have in common with the Castor and Pollex by "reifying" (making something into a "thing") relations. Relations therefore are a special sort of universal. They can be exemplified wholly and completely in multiple pairs simultaneously.

Relations have several important properties to which attention should be drawn. First, in some relations the order of the pairs matters. Consider the *less than* relation. There are many pairs <*x,y*> such that *x is less than y.* For example, <1,2>, <5,7>, <36,215>. These all share the feature that the first is less than the second. But they cannot be revered. In technical jargon, the less-than relation is asymmetric. That is, if *x is less than y*, we know *it is not the case that y is less than x.*

Here <*x,y*> is called an *ordered-pair,* and it is stipulated by convention that the order makes a difference. That is, the pair <*x,y*> is always different from the reversed pair <*y,x*> except in the unusual case in which *x* and *y* are one and the same individual. It is ordered pairs then that share a relation and "have it" in common.

Second, relations that hold between pairs, the so-called *two-place* relations, are not the only sort of relation. There are also relations that hold among triples. For example, it takes three things (in order) for cases of *betweeness* to happen. Utah is between Nevada and Colorado, Cincinnati is between Dayton and Lexington. George II is between George I and George III. These are *three-place* relations. In principle there are also four-place relations that hold among ordered quadruples. Likewise there are *n-place* relations exemplified by sets of ordered grouping of *n* objects, for any number *n.* As in the case of two-place relations, the order continues to matter. If *x* is between *y* and *z,* then *y* cannot be between *x* and *z.* Thus, an *n-place relation* is the commonality shared by what are called ordered *n*-tuples, which are represented by the notation <x_1,…,x_n>.

Third, relations are tied to characteristic grammatical forms. Two place relations between *x* and *y*, for example, are typically expressed in English by subject-verb-object sentences, like *x loves y* and *x teaches y.* Two place relations are also expressed by sentences that link a subject to an "oblique object" by intransitive verb and a prepositions, for example *x talks to y* and *x sits under y.* Comparative adjectives also link two relata: *x is taller than y, x is less than y,* and *x is sillier than y.* Possessive expressions link two objects: *x is the brother of y* and *x is the creator of y.* All these syntactic forms share the feature that they link two proper noun phrases. Three place relations link three proper noun phrases, like *x is between y and z, x talked to y about z,* and *x saw y* *sitting on z.* In general, an open sentence P with n free variables $x_1,\ldots,x_n$, i.e. $P(x_1,\ldots,x_n)$, can be used to describe what is shared by a group of ordered n-tuples $<x_1,\ldots,x_n>$.

Lastly, like properties but even more so, there has been controversy in philosophy about whether relations really exist, and if so, how the behave. These controversies are settled in set theory by the simple technique. First the notion of an ordered n-tuple is in terms of sets. Then n-place relation is defined as a set of n-tuples.

The first task of defining $<x_1,\ldots,x_n>$ as a set is accomplished by finding some definition – it does not matter which – so long as it insures that $<x_1,\ldots,x_n>$ different from $<y_1,\ldots,y_n>$ except in the unusual case in which each x_i is identical to y_i. The definition that works is not very intuitive, but it works.

Définition

$<x,y> \;=\; \{x,\{x,y\}\}$

$<x_1,\ldots,x_{n+1}> \;=\; <<x_1,\ldots,x_n> x_{n+1}>$

Theorem. $<x_1,\ldots,x_{n+1}> \;=\; <y_1,\ldots,y_{n+1}>$ iff for all i=1,…,n, $x_i = y_i$.

An n-place relation is a set of n-tuples. If $P(x_1,\ldots,x_n)$ is an open sentence, then there is a relation R (a set) such that the n-tuple $<x_1,\ldots,x_n>$ is in R if and only if $x_1,\ldots,x_n$ in order satisfy $P(x_1,\ldots,x_n)$. One way to name the relation R is by using the notation for a set abstract:

Notation

$\{<x_1,\ldots,x_n>|\ P(x_1,\ldots,x_n)\}$ read *the set of all n-tuples* $<x_1,\ldots,x_n>$ *such that* $P(x_1,\ldots,x_n)$.

Since relations are special cases of sets the axioms of set theory apply to them. It follows as a theorem that an n-place relation exists if it can be defined by an open sentence $P(x_1,\ldots,x_n)$.

Theorem. For any open sentence $P(x_1,\ldots,x_n)$,

1. there exists an R such that for any $x_1,\ldots,x_n$,
   $<x_1,\ldots,x_n>\in R$ iff $P(x_1,\ldots,x_n)$.
2. there exists a set $\{<x_1,\ldots,x_n>|\ P(x_1,\ldots,x_n)\}$ such that for any $y_1,\ldots,y_{n+1}$,
   $<y_1,\ldots,y_{n+1}>\in\{<x_1,\ldots,x_n>|\ P(x_1,\ldots,x_n)\}$ iff $P(y_1,\ldots,y_n)$.

Because relations are sets and obey the Principles of Abstraction and Extensionality, relations exist if they are definable. Because relations are sets and obey the Principles of Extensionality, two relations are identical if they have the same members, i.e. if they are true of the same n-tuples.

Theorem. If R and S are n-place relations, then

R=S iff for any $x_1,\ldots,x_n$, $<x_1,\ldots,x_n>\in R$ iff $<x_1,\ldots,x_n>\in S$.

There is a special sort of relation called a ***function*** or, equivalently, an ***operation***. An n+1-place relation relates by definition n+1 objects. If it does so in the special way that given the first *n* we can uniquely determine the $n+1^{st}$, then the relation is a function.

Take the simple case of the 2-place relation *is fathered by.* This relation, call it *f,* is a function because given the first member of a pair, a child, the relation pairs with it a unique second member of a pair, the child's father. The fact that there is one and only one entity paired with the first element of a pair in *f* permits a new notation. The fact that Able is fathered by Adam in the fathered by function *f* in expressed as <Able,Adam>$\in f$. This may be rewritten as *f*(Able)=Adam. The first element of the pair is called the function's ***argument***, and the second its ***value***. A function is a relation that pairs with each argument a unique value. Hence *f*(Able)=Adam is read the value for the function *f* for the argument Able is Adam. The set of all arguments (inputs, left members of pairs) is called the function's ***domain***. The set of all values (outputs, second members of pairs) is called its ***range***.

Note that some two-place relations, like *is the brother of,* are not functions because they contain two pairs with the same argument, e.g. <Able,Cain> and <Able,Seth>. There is no unique entity that is "the" brother of Able. In arithmetic *has as its absolute value* is a function, though contains both <-2,2> and <2,2>, every argument has a unique value, |-2|=2 and |2|=2. However *has as square root* is not a function because it contains both <4,2> and <4,-2>.

Likewise a 3-place relation, which is a set of triples, is a function if the ordered pair that makes up the first two parts of a triple uniquely determines the third element of the triple. In arithmetic, for example, <1,5,6>, <5,1,6> <3,3,6>, <4,2,6>,<2,4,6> are all in the three-place relation +. Accordingly, we express <1,5,6>$\in$ + by the functional notation +(1,5)=6, or in its more usual "infix" form, by 1+5=6. In an *n*+1-place relation an argument is any *n*-tuple that makes up the first *n*-places of any of the *n*+1-tuples it contains, and a value is any object that occupies the $n+1^{st}$ place. The domain is the set of all arguments and the range is the set of all values. Because any *n*+1-place relation *f* that is a function has exactly *n*-places in any of its arguments, it is also called an ***n-place*** function and we write $<x_1,\ldots,x_{n+1}>\in f$ as $f\,(x_1,\ldots,x_n)= x_{n+1}$.

We shall be encountering functions primarily is two contexts. First they will occur with selected sets and relations as components of what are called abstract structures. A structure is a mini-world in which categories of entities are classified and related in specified ways, and in which some entities are linked as functions. In these worlds functions may be viewed as input-output processes or as rule-like production procedures. Structures of these sorts will differ in systematic ways reflected in the general laws governing their components. The structures we shall study will exhibit infinity in its various forms. The second way we shall use functions is to count. A so-called 1 to 1 correspondence is a function that maps each entity in the domain to one and only one entity in the range and is such that no entity of the range is left unpaired with an entity in the domain. We shall use such correspondences to "count" by pairing objects in a set 1 to 1 a set of counting numbers.

Definitions

AxB	Cartesian product of A and B	$\{<x,y>\mid x\in A \& y\in B\}$
A^2	Cartesian product of A and A	AxA
A^n	Cartesian produce of $A_1,\ldots,A_n$	$A_1x\ldots xA_n$
V^2	The universal (binary) relation	VxV
$\mathfrak{P}(V^2)$	the set of 2-place relations	
$\mathfrak{P}(V^n)$	the set of n-place relations	

R is a ***binary relation***		$R\subseteq V^2$
R is a ***n*-place relation**		$R\subseteq V^n$
f is a ***1-place function***	f is a binary relation and f is "functional": $\forall x\forall y\forall z((<x,y>\in f \ \& <x,z>\in f)\rightarrow y=z)$	
f is a ***n+1-place function***	f is an n-place relation and $\forall x_1\ldots\forall x_n\forall y\forall z((<x_1,\ldots,x_n,y>\in f \ \& <x_1,\ldots,x_n,z>\in f)\rightarrow y=z)$	

If f is a 1-place function, $f(x)=y$ *means* $<x,y>\in f$
If $f(x)=y$, then x is an argument of f and y is a value.
Domain(f) = $\{x\mid \exists y f(x)=y\}$
Range(f) = $\{y\mid \exists y f(x)=y\}$
$f^{-1} = \{<y,x,>\mid f(x)=y)\}$ f^{-1} is called the ***inverse*** of f

If f is a n-place function, $f(x_1,\ldots,x_n)=y\}$ means $<x_1,\ldots,x_n,y>$
If $f(x_1,\ldots,x_n)=y$, then $<x_1,\ldots,x_n>$ is an argument of f and y is a value.
Domain(f) = $\{<x_1,\ldots,x_n>\mid \exists y f(x_1,\ldots,x_n)=y\}$
Range(f) = $\{y\mid \exists y f(x_1,\ldots,x_n)=y\}$

$f(A\xrightarrow{\text{into}}B)$	f is a 1-place function, Domain(f), and Range(f)$\subseteq B$
$f(A\xrightarrow{\text{onto}}B)$	f is a 1-place function, Domain(f), and Range(f)$=B$
$f(A\xrightarrow{\text{1-1 onto}}B)$	f is a 1-place function, Domain(f), Range(f)$=B$, and f^{-1} is a 1-place function
f is a partial function on A	Domain(f)$\subseteq A$

6. Axiomatic Set Theory

The axioms of naïve set theory are seemingly simple and self-evidently. However, Bertrand Russell discovered at the turn of the 20th century that these axioms entail a contradiction that bears his name. By convention a contradiction that cannot be explained because it seems to follow from premises that are true and self-evident is called a ***paradox***. Russell's contradiction is a paradox because it is entailed by the axioms of naïve set theory that appear to be simple and true.

Theorem (Russell's Paradox). The Principle of Abstraction is false.

Proof. Let us assume the Principle of Abstraction. By the Principle (applying it to the open sentence '$x\notin x$'):

(1) For some A, for any x, $x\in A$ iff $x\notin x$.

Let us consider one such A.

(2) for any x, $x\in A$ iff $x\notin x$.

Since this is true of any *x*, it is true of *A*:

(3) $A \in A$ iff $A \notin A$.

But this is a contradiction. Hence the Principle of Abstraction is false. **QED.**

Since the two axioms of naïve set theory are contradictory, they somehow contain a falsehood. Logicians speculate that the problem lies in the fact that the Principle of Abstraction allows for the existence of sets that are too large. On this line of thinking, the Principle errs in positing the existence of any definable set whatever, even sets that are not built up from previously members. They may even be circular in that the set itself appears is a member of itself or a member of sets that in turn are in it. It is speculated that if a set is built up in a careful way from elements that we already know exist will not contain a contraction. In the branch of logic known as ***axiomatic set theory*** the Principle of Abstraction is replaced by a series of axioms that insure the existence of restricted categories of progressively larger sets. They start with a severely restricted version of the Principle of Abstraction that says that it is possible to define any set you like but only on the condition that this set is to be separated as a subset from another set that is previously defined and known to exist. The new axiom is therefore called the ***Axiom of Separation***. Subsequent axioms insure the existence of additional categories of sets: the union of two previously defined sets exist, the ordered-pair (a kind of set) of two elements exists, the power set (i.e. the set of subsets) of a set exists, and a subset of a function's range exist. The last axiom is the most powerful and generates the "largest" sets. It says that given any family of sets, you can form a new set by choosing one representative element from each set in the family. Accordingly it is called the ***Axiom of Choice***. To state the axioms two definitions are helpful.

Definitions.

1. If *f* is a function and *A* is a set, then ***the image of A* under *f*,** briefly f"A, is $\{x \mid \exists y\ (y \in A \ \&\ f(x)=y)\}$. (In English, f"A is the set of all values of *f* for arguments in *A*.)
2. If *F* is a family of sets, then *A* is a ***choice set*** for *F* iff for any $B \in F$, there is one and only one element *x* of *A* such that $x \in B$.

Zermelo-Frankle (ZF) Axioms for Set Theory

1. **Axiom of Separation**. Let $P(x)$ be an open sentence.
   For any *A*, there is a set *B* such that $B \subseteq A$, and for all *x*, $x \in B$ then iff $P(x)$
2. **Union Axiom**. For any *A* and *B*, the union $A \cup B$ of *A* and *B* exists.
3. **Pair Axiom**. For any *x* and *y*, the ordered-pair $<x,y>$ of *x* and *y* exists.
4. **Power Set Axiom**. For any *A*, the power ser $\mathcal{P}(A)$ of *A* exists.
5. **Axiom of Infinity**. An infinite set exists.
6. **Axiom of Replacement**. For any *A* and any function *f*, the image f"A of *A* under *f* exists.
7. **Axiom of Choice**. For any family of sets *F*, some choice set of *F* exists.

The Axiom of Choice may be formulated in equivalent ways, two of which are relevant to these notes:

The Well Ordering Principle. For any set *A* there is some relation $\leq$ such that $<A,\leq>$ is a well ordering.

Zorn's Lemma. If every chain C (i.e. for any x and y of C, either $x \leq y$ or $y \leq x$) of a partially ordered structure $<X,\leq>$ has an upper bound, then $<X,\leq>$ has a maximal element.

Because the Axiom of Choice is the most powerful of the existence axioms of ZF and is somewhat less obvious, it is more controversial than the others. Its consistency with and independence of the other axioms has therefore been the subject of investigation. Here are the two of the most famous results in the field.

Theorem (Gödel). The negation of the Axiom of Choice is consistent with the earlier axioms of ZF.

Theorem (Cohen). The Axiom of Choice cannot be proven from the earlier axioms of ZF.

In the notes any theorem that requires the Axiom of Choice for its proof is prefixed with *.

7. Abstract Structures

We all have a good intuitive idea of a "structure." Examples include buildings, governmental institutions, ecologies, and polyhedral. In the branch of mathematics known as ***abstract*** or ***universal algebra*** the general properties of structures are studied, and these ideas help explain the structures we find in logic like those of grammars, semantic interpretations, and inferential systems.

The raw intuition behind the mathematical definition of a structure is that of an architect's blueprint. The blue print succeeds in describing a building by first listing its various materials and then by a diagram describing the relations that must obtain among these "building blocks" in the finished structure. In algebra a structure is defined in a similar way. First a list of set $A_1,...,A_k$ is given. These may be viewed as list of building blocks divided into various kinds or classes. Next are listed the relations $R_1,...,R_i$ and functions $f_1,...,f_m$ that hold among these materials. (Recall that functions are just a sub-variety of relations.) Lastly it is useful to list some specific individual building blocks $O_1,...,O_m$ that have special importance in the structure. It is customary to list all the elements of the structure in order, *i.e.* as an ordered tuple: $< A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_m >$.

Definition. An *abstract structure* **is any $<A_1,...,A_k,R_1,...,R_l,f_1,...,f_m,O_1,...,O_n>$ such that:**

for each i=1…k, A_l is a set,
for each i=1…l, R_i is a relation on $C = \cup\{A_1,...,A_n\}$,
for each i=1…m, f_i is a function on $C = \cup\{A_1,...,A_n\}$, and
for each i=1…n, $O_i \in C = \cup\{A_1,...,A_n\}$.

It is also common to investigate a family of structures with similar properties, and to assign the family a name, e.g. group, ring, lattice, or Boolean algebra. The properties defining such a family are usually formulated as defining conditions on the type of sets, relations, functions and designated elements that fall into the family. Sometimes these restrictions are referred to as the "axioms" of the structure-type. Strictly speaking they are not part of a genuine axiom system. Rather they are clauses appearing in the abstract definition of a particular set (family) of structures. Let us review some familiar examples.

The familiar "less than" relation on numbers, symbolized by $\leq$, and the subset relation on sets, symbolized by $\subseteq$, are instances of what is known as partial ordering. In algebra such orderings are viewed as structures. To define such a structure, however, we must first define some standard properties of relations. We then define several common varieties of ordered-structures.

Definitions. Properties of Relations. A binary relation $\leq$ is said to be:

1. ***reflexive*** iff for any x, $x\leq x$;
2. ***transitive*** iff for any x,y, and z, if $x\leq y$ and $y\leq z$, then $x\leq z$;
3. ***symmetric*** iff for any x and y, if $x\leq y$ then $y\leq x$;
4. ***asymmetric*** iff for any x and y, if $x\leq y$ then not ($y\leq x$);
5. ***antisymmetric*** iff for any x and y, if $x\leq y$ and $y\leq x$, then $x=y$;
6. ***connected*** iff $\leq$ for any x and y, either $x\leq y$ or $y\leq x$;
7. x is a ***$\leq$-least element*** of B iff $x\in B$ and for any $y\in B$, $x\leq y$.
8. x is a ***$\leq$-greatest element*** of B iff $x\in B$ and for any $y\in B$, $y\leq x$.
9. x is a ***$\leq$-greatest lower bound*** (***infimum***) of B iff x is a lower bound of B, and for any y, if y is a lower bound of B, then x$\leq$y.
10. x is a ***$\leq$-least upper bound*** (***supremum***) of B iff x is an upper bound of B, and for any y, if y is an upper bound of B, then $y\leq x$.

Definitions. Any structure $<B,\leq>$ such that B is a non-empty set and $\leq$ is a binary relation on B is called:

1. a ***pre-*** or ***quasi-ordering*** iff $\leq$ is reflexive, transitive;;
2. a ***partially ordering*** iff $\leq$ is a pre-ordering and antisymmetric;
3. a ***total*** or ***linear*** ordering iff $\leq$ is partial and connected;
4. a ***well-ordering*** iff, $\leq$ is a total ordering and for any subset C of B, B has a $\leq$-least element.
5. a ***dense ordering*** iff it is a total ordering and for any x and y in B such that $x\leq y$, there is a z in B such that $x\leq z$ and $z\leq y$.
6. a ***continuous ordering*** if it has the ***Dedekind completeness property***, i.e. every nonempty subset C of B that has an $\leq$-upper bound has a $\leq$-least upper bound.

Definitions. Properties of Binary Operations (aka Functions). Let $\bullet$ and $\blacklozenge$ be binary operations on a set B, and let us write $\bullet(x,y)$ using "infix" notation as $x\bullet y$. Likewise for $\blacklozenge(x,y)$. Then,

1. B is ***closed under*** $\bullet$ iff for all x,y of B, $x\bullet y\in B$,
2. $\bullet$ is ***associative*** iff for all x,y of B, $x\bullet y=y\bullet x$,
3. $\bullet$ is c***ommutative*** iff for all x,y of B, $x\bullet(y\bullet z)=(x\bullet y)\bullet z$,
4. $\bullet$ is ***idempotent*** iff for all x,y of B, $x\bullet x=x$,
5. e is an ***identity element*** for $\bullet$ iff for all x of B, $x\bullet e = x$,
6. e is an ***inverse element*** relative to an identity element i of B iff for all x of B, $x\bullet i=e$
7. $\bullet$ and $\blacklozenge$ are distributive for all x,y,z of B,
   i. $x\bullet(y\blacklozenge z) = (x\bullet y)\blacklozenge(x\bullet z)$, and
   ii. $(x\blacklozenge y)\bullet z = (x\bullet z)\blacklozenge(y\bullet z)$

Definitions. Types of Ordered Structures.

1. A structure $<B,\wedge>$/$<B,\vee>$ is a ***meet/join semi-lattice*** iff $\wedge/\vee$ is a binary operation under which B is closed and $\wedge/\vee$ is associative, commutative, and idempotent.

2. If $<B,\wedge>$ is a meet semi-lattice, then ***the ordering relation ≤ on*** B is defined as
   $x\leq y$ iff $x\wedge y=x$.
3. If $<B,\wedge>$ is a join semi-lattice, then ***the ordering relation ≤*** *on* B is defined as
   $x\leq y$ iff $x\vee y=y$.
4. The structure $<B,\wedge,\vee>$ is a **lattice** iff $<B,\wedge>$ and $<B,\vee>$ are receptively meet and join semi-lattices, and the ordering relation ≤ on B is defined as: $x\leq y$ iff $x\wedge y=x$ iff $x\vee y=y$.
5. If $<B,\wedge,\vee>$ is a lattice, then 0 is ***the least element*** of B iff
   $0\in B$
   for any x in B, $0\leq x$,
   $0\wedge x=0$ and
   $0\vee x=x$.
6. If $<B,\wedge,\vee>$ is a lattice, then 0 is ***the greatest element*** of B iff
   $1\in B$
   for any x in B, $x\leq 1$,
   $1\wedge x=x$ and
   $1\vee x=1$.
7. If $<B,\leq>$ is a partially ordered structure and x and y are in B, then ***the greatest lower bound*** (briefly, ***glb***) of $\{x,y\}$ (if it exists) is the $z\in B$ such that
   $z\leq x$ and $z\leq y$
   for any w in B if $w\leq x$ and $w\leq y$, then $w\leq z$.
8. If $<B,\leq>$ is a partially ordered structure and x and y are in B, then ***the least upper bound*** (briefly, ***lub***) of $\{x,y\}$ (if it exists) is the $z\in B$ such that
   $x\leq z$ and $y\leq z$
   for any w in B if $x\leq w$ and $y\leq w$, then $z\leq w$.
9. A lattice $<B,\wedge,\vee>$ is **distributive** iff
   $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$, and
   $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$.
10. If $<B,\wedge,\vee,0,1>$ is a structure such that $<B,\wedge,\vee>$ is a lattice and 0 and 1 are respectively its least and greatest elements, then $-$ is a (***unique***) ***complementation operation*** on the structure iff
    $-$ is a one-place operation on B $\quad -1=0$
    for any $x\in B$, $-x\in B$ $\quad -0=1$
    $x\wedge -x=1$ $\quad -(x\wedge y)=-x\vee -y$
    $x\vee -x=0$ $\quad -(x\vee y)-x\wedge -y$
    $-x=x$ $\quad x\leq y$ iff $-x\wedge y=0$ iff $-y\leq -x$ iff $-x\vee y=1$
11. A structure $<B,\wedge,\vee,-,0,1>$ is a ***Boolean algebra*** iff
    $<B,\wedge,\vee>$ is a lattice
    $<B,\wedge,\vee>$ is distributive
    0 and 1 are respectively the least and greatest elements of $<B,\wedge,\vee>$
    $-$ is a complementation operation on $<B,\wedge,\vee,0,1>$

Theorems.

1. If $<B,\wedge,\vee>$ is a lattice, then $<B,\leq>$ is a partial ordering.
2. If $\leq$ is a partial ordering on a set B and if for any x and y in B, the glb$\{x,y\}$ and the lub$\{x,y\}$ exist and are in B, and if $\wedge$ and $\vee$ are binary operations on B defined as follows
   $x\wedge y$ = glb$\{x,y\}$, and $x\vee y$ = lub$\{x,y\}$,
   then the structure $<B,\wedge,\vee>$ is a lattice with ordering relation $\leq$.

8. Sameness of Structure

One of the most important ideas in algebra is sameness of structure. Two teacups from the same set and two pennies have the same structure. So too do two twins. In these cases the structures match very closely. But family members and even members of the same species have some features of structure in common. More abstractly, the reason maps work is that there is a similarity of structure between geographical features in the world and the symbols on the map that represent them. Blueprints work for this reason too. Mathematically this sameness is explained by saying that there is a mapping from the entities of one structure into the entities of a second in such a way that the mapping "preserves structure." Informally, if we have two structures and entity x_1 in the first that "corresponds" to an entity x_2 in the second, we may call x_2 ***the representative*** of x_1. Often one structure may be more complex than the other, yet both exhibit some structural features in common. One way this happens occurs when elements of the more complex are "identified" or viewed as a unit in the second. This happens, for example, in our representative democracy in which a single individual in Congress represents all the citizens in an election district. Thus for a "similarity of structure" to obtain we require as a minimum that each entity of one structure corresponds to one and only one entity in the second. In mathematical terms, there is an ***into-function*** that assigns a ***value*** in the second structure to each ***argument*** in the first. If h is the mapping function, then $h(x_1)=x_2$. Here $h(x_1)$ is the representative of x_1. Such a mapping is called a ***homomorphism*** (from the Greek *homos = the same* and *morphos=structure*.)

Definition. Two structures $S=<A_1,...,A_k,R_1,...,R_l,f_1,...,f_m,O_1,...,O_n>$ and $S'=<A'_1,...,A'_k, R'_1,...,R'_l,f'_1,...,f'_m,O'_1,...,O'_n>$ are said to be of the ***same character*** or ***type*** iff

for each i=1,…,l, there is some n such that R_i and R'_i are both n-place relations, and

for each i=1,…,n, there is some n such that f_i and f'_i are both n-place functions.

Very often a discussion is clearly limited to structures of the same type. When this restriction is clear, it is tedious to keep mentioning it, and it is usually assumed without saying so explicitly.

Definition. If $S=<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n>$ and $S'=<A'_1,...,A'_k,R'_1,...,R'_i, f'_1,...,f'_m,O'_1,...,O'_n>$ are structures of the same character, h is called a ***homomorphism from*** S to S' iff h is a function from $\cup\{A_1,...,A_n\}$ into $\cup\{A'_1,...,A'_n\}$ such that

1. for each i=1,…,k, if $x\in A_i$, then $h(x_i)\in A'_i$;
2. for each i=1,…,l, $<x_1,...,x_n>\in R_i$ iff $<h(x_1),...,h(x_n)>\in R'_i$;
3. for each i=1,…,m, $h(f_i(x_1,...,x_n) = f'_i(h(x_1),...,h(x_n))$;
4. for each i=1,…,n, $h(O_i)=O'_i$.

9. Sameness of Kind

Sameness is one of the "great ideas." Aristotle was the first to clearly distinguish ***numerical identity*** (he coined the term) from other sorts of sameness. Two entities are numerically identical if they are literally one and the same thing. Sometimes we say that things that are identical in this sense are "the same". But sameness is also used in ordinary speech for a less discriminating relation that in effect classifies entities into groups that are same in some relevant sense though not literally identical. This is ***sameness of kind*** or ***similarity***. In algebra "sameness" relations, including both numerical identity and similarity, as "equivalence relations".

Definition. A binary relation $\equiv$ on a set A is said to be an ***equivalence relation*** on A iff $\equiv$ is reflexive, transitive and symmetric. The ***equivalence class*** of x under $\equiv$, briefly $[x]_{\equiv}$, is defined as $\{x| x\equiv x\}$.

Clearly numerical identity counts as an equivalence relation, but so do many other relations of "sameness."

Sameness of kind is also discussed in terms of sets. One way to show things are the same is to sort them into mutually exclusive sets. Informally we do this if we say what property defines each set, e.g. the set of red things, the set of blue things, etc. We often do this in mathematics by means of "set abstracts." We may go through everything there is and find such defining characteristics for "kinds" or "sorts," and classify then into sets $\{x| P_1(x)\},...,\{x|P_n(x)\}$. If the sets are mutually exclusive and exhaustive (i.e. leave nothing out) they are called a ***partition*** of *A.*

Definition. A family $F=\{B_1,...,B_n\}$ of sets is a ***partition*** of a set A iff, $A=\cup\{B_1,...,B_n\}$ and no two B_i and B_j overlap (i.e. for each i and j, $B_i \cap B_j = \varnothing$).

There is moreover a way to generate a partition from a sameness relations and vice versa.

Theorems

1. If a family $F=\{B_1,...,B_n\}$ of sets is a partition of a set A, then the binary relation $\equiv$ on A is defined as follows: $x\equiv y$ iff for some i, $x\in A_i$ and $y\in A_i$ is an equivalence relation.
2. The family of all equivalence classes $[x]_{\equiv}$ for all x in a given set A is a partition of A.

The set of all entities from the first structure that have the same representative are in a sense "the same:" they form an equivalence class. For example, the set of citizens represented by the same congressman is a equivalence class. One of direct consequences of these ideas is the fact that equivalence classes do not overlap and that they exhaust all the entities of the first structure.

Theorem

Let h be a homomorphism from $S=\langle A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n\rangle$ to $S'=\langle A'_1,...,A'_k,R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n\rangle$, and let the binary relation $\equiv_h$ on $C=\cup\{A_1,...,A_n\}$ be defined as follows:

$x \equiv_h y$ iff $h(x)=h(y)$.

It follows that:

1. $\equiv_h$ is an equivalence relation on C.

Furthermore, if $[x]_h$, called ***the equivalence class*** of x under h is defined as $\{y|\ y \equiv_h x\}$, then it follows that:

2. the family F of all equivalence classes, i.e. $\{[x]_h \mid x \in C\}$, is a partition of C.

10. Identity of Structure

If a structural representation is so tight that it exhausts the elements of the second structure in the sense that all of its elements are representatives of some entity in the first, then the representation function is said to be ***onto***. There are, for example, no voting members of Congress that do not represent some state. In Germany, however, where some members of Parliament are allotted to parties due to national voting percentages there are members that do not represent a specific district.

In some instances the representation is so fine grained that no two entities of the first structure have the same representative. Such a mapping would be too cumbersome for Congress, but it is essential for social security numbers. Such mappings are said to be 1 to 1. Any mapping that is 1 to 1 and onto totally replicates the structure and entities of the first structure. It is called an ***isomorphism*** (from *isos*=*equal*).

Definition. If h is a homomorphism from $S=<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n>$ to $S'=<A'_1,...,A'_k,R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n>$, then h is said to be an ***isomorphism*** from S to S' if h is a 1-1 and onto mapping.

It follows from the definitions that given a homomorphism from a first structure to a second we can define a third structure made up of the equivalence classes of the first and this new structure can be made to have exactly the same structure as (be isomorphic to) the second. This new structure is called the quotient algebra.

Definition. If h is a homomorphism from $S=<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n>$ to $S'=<A'_1,...,A'_k,R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n>$, then ***the quotient algebra*** for $<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n>$ under h is $S''=<A''_1,...,A''_k,R''_1,...,R''_i,f''_1,...,f''_m,O''_1,...,O''_n>$ defined as follows:

given $x \equiv_h y$ iff $h(x)=h(y)$ and $[x]_h$ to be $\{y|\ y \equiv_h x\}$,

$A''_i = \{[x]_h \mid x \in A_i\}$

$<[x_1]_h,\ldots,[x_n]_h> \in R''_i$ iff $<x_1,...,x_n> \in R_i$

$f_i([x_1]_h,\ldots,[x_n]_h)= [f_i(x_1,...,x_n)]$

$O''=[O_i]_h$

Theorem. If h is a homomorphism from $S=<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n>$ to $S'=<A'_1,...,A'_k,R'_1,...,R_i,f'_1,...,f'_m,O'_1,...,O'_n>$,
then S is homomorphic to its quotient algebra S'' under h , and S' is isomorphic to S''.

11. Computation as an Abstraction from of Numbers

There are various ways in which "the infinite" is studied by the use of abstract structures. We shall begin with one of the most basic. The construction of numbers. Infinity, at least in one sense, is a feature of counting. Counting uses numbers. A group is infinite if counting it with numbers never stops. Infinity is also a property of divisions. Spatial distances and temporal intervals appear to be divisible into smaller and smaller parts forever. One way these ideas have been studied in modern mathematics is by the definition of structure that exhibits the relevant properties. In this section we define a series of structure with progressively more structure. The simplest called a *semi-group* has scarcely any structure at all, only am associative single binary operation. The most complex, that of the real numbers, has a wide variety of relations and operations. It allows for the counting of infinite sets, and even of sets that are bigger than countably infinite. It also allows for the sort of infinite divisibility typical of distances and intervals. We begin by defining some rather structures with relations and operations that exhibit features common to those of ordinary arithmetic.

Definitions

1. A ***semi-group*** is a structure $<B,\bullet>$ such that $\bullet$ is an associative binary operation on B.
2. A ***monoid*** is a structure $<B,\bullet,1>$ such that $<B,\bullet>$ is a semi-group and 1 is an identity element on $<B,\bullet>$ relative to $\bullet$.
3. A **(*commutative*) *ring*** is a structure $<B,+,\bullet,0,1>$ such that
   i. + and $\bullet$ are binary operations on B that are associative and commutative,
   ii. 0 is an identity element relative to $\bullet$ (the ***additive identity***) and
   iii. 1 is an identity element relative to $\bullet$ (the ***multiplicative identity***).
   iv. for any x in B, there exists a y in B (***the additive inverse*** of x) such that $x + y =0$
   v. + and $\bullet$ are distributive in B
4. The ***subtraction operation*** – relative to a commutative ring $<B,+,\bullet,0,1>$ – is a binary operation on B defined as follows: for any x and y of B, $x–y = x + (–y)$. Hence a ring $<B,+,\bullet,0,1>$ may be identified with the structure $<B,+,\bullet,–,0,1>$ such that – is the subtraction operation on $<B,+,\bullet,0,1>$.

Theorem. Every for every element x of a ring $<B,+,\bullet,0,1>$ there is an + inverse element called $–x$.

Definitions

1. An ***ordered ring*** is a structure $<B,\leq,+,\bullet,–,0,1>$ such that
   i. $<B,+,\bullet,–,0,1>$ is a ring,
   ii. is a total ordering on B,
   iii. for any x, y, and z of B, if $x \leq y$ then $x + y \leq y +z$, and
   iv. for any x, y, and z such that $0\leq z$ of B, if $x \leq y$ then $z \bullet x \leq z \bullet$ y.
2. An ***identity relation*** relative to a ring $<B,+,\bullet,0,1>$ is a binary relation = on B defined as follows: $0\neq1$ and for any x in B such that $x\neq0$, there is a y in B such that $x \bullet y = 1$
3. A ***field*** is a structure $<B,=,+,\bullet,–, 0,1>$ such that $<B,+,\bullet,–,0,1>$ is a ring and = is an identity relation on $<B,+,\bullet,0,1>$.
4. An ***ordered field*** is a structure $<B,\leq,=,+,\bullet,–, 0,1>$ such that $<B,\leq,+,\bullet,–,0,1>$ is an ordered ring and $<B,=,+,\bullet,–, 0,1>$ is a field.

Extensive Measurement

Among the elements of a set of physical objects or quantities – sets that are not mathematical entities like numbers -- comparisons (e.g. *is taller than, is more beautiful than, is better than*) are made by non-mathematical means. We use relations that are defined in physical, non-mathematical terms, but which nevertheless amount to orderings. Typically, these are named in natural language by ***comparative adjectives***, which are often associated with sets of what are called by linguists ***scalar adjectives***. Scalar adjectives are used to name regions of various ranks in an ordering. For example, the comparative adjective *happier than* describes a physical ordering in nature. Associated with it is the family of scalars adjectives *ecstatic, happy, content, so-so, sad, unhappy, miserable*.

We have seen how a relation $\leq$ may posses a more or less rigorous structure, ranging from pre-orderings to the dense continuous orderings of the real numbers. Here we shall investigate the standard features of natural structures that make them amenable to numerical measurements open that allow arithmetical computations on the measurement values by means of the standard arithmetical operations like addition and multiplication.

First let us do some ontology to figure out exactly what sort of thing is being ordered when we order quantities. Socrates and Plato are individual human beings, things in the world, who possess properties. Some of these properties admit of comparisons of "more" or "less," as in *Socrates is taller than Plato, Plato is more handsome than Socrates*. Some admit of numerical comparisons by so-called measure phrases, as in *Socrates is five inches taller than Plato.* Ontologically what is being ordered are the something like the extent to which some thing possesses a property, and what is being measured numerically is a quantity. Language possess mass nouns formed from adjectives for this purpose. What is it that Socrates has more of than Plato? Tallness. How much more tallness does he have? Five inches. What is ranked by the relevant ordering relation associated with a comparative adjective is "quantities" of the "mass" indicated by its associated mass noun.

For the purpose of clarifying the conditions necessary for the numerical measure of "masses" of this sort, social scientists make use of a special operation binary operation $\blacklozenge$ defined on quantities of a given "mass." If we let x and y represent two quantities of a given mass, like heat, weight, or length, we represent the quantity obtained by combining x and y by $x\blacklozenge y$. If $\blacklozenge$ is defined for weight, $x\blacklozenge y$ could be defined as the combined distance on a scale of the two distances x and y. If it is defined for volume, $x\blacklozenge y$ could be defined as the amount of water displaced first by x and then by y. The capture the fact that $x\blacklozenge y$ joins together two quantities, it could be called "combination", but its standard name in measurement theory is ***concatenation***.

Mathematically the challenge for measurement theory is to state the conditions under which an ordering relation $\leq$ on units of mass allows for those units to be quantified in such a way that numerical operations like addition and multiplication are meaningfully applicable to them. It is possible to state these conditions in terms of the properties of concatenation.[1] For this purpose we define an abstract structure composed of a set *B* (which are intuitively "quantities," "masses", or "extensions"), an ordering relation $\leq$ on these masses, and a concatenation operation $\blacklozenge$ that combines them. We also define the idea *nx* of concatenating the mass quantity *x* with itself *n*-times. Intuitively 2*x*, which is the same as $x\blacklozenge x$, is "twice" the "mass" of *x.*

$<B,\leq,\blacklozenge>$ is a ***(positive) closed extensive structure*** iff for any $x,y,u,v \in B$:

1. $\leq$ is a weak ordering on *B:* $\leq$ is transitive and connected.
2. $\blacklozenge$ is associative: $x\blacklozenge y = y\blacklozenge x.$
3. $\leq$ is monotonic: $x\leq y$ iff $(x\blacklozenge u)\leq(y\blacklozenge u)$ iff $(u\blacklozenge x)\leq(u\blacklozenge y)$.
4. $\blacklozenge$ is positive: for any $x,y\in B$ $x \leq x\blacklozenge y$. (Hence $x \leq x\blacklozenge x$).
5. $\leq$ is Archimedean: Let *nx* (read ***the concatenation of** x n **times***) be defined as follows (a) $1x=x$, and (b) $(n+1)x=nx\blacklozenge x.$ If $x<y$, then for any $u,v\in B$, there exists a positive integer *n* such that $(nx\blacklozenge u) \leq (ny\blacklozenge v)$.

Condition 1 is a minimal condition for considering $\leq$ to be an "ordering." Conditions 2-5 insure that concatenation provides the basis for mapping $<B,\leq,\blacklozenge>$ onto a structure of numbers so the mapping, a "measure" assignment, showing that $<B,\leq,\blacklozenge>$ reproduces numerical structure.

Intuitively, Condition 4 is the most difficult to understand. Intuitively, it says that no matter how big a head start (*u*) you give a lesser extension (*x*), you can always find enough units, namely *n,* of the larger extension *y* (possibly enlarged by *v*) so that *n* units of *y* together with *v* will be bigger than *n* units of *x* with its the head start *u.*

There is a less general by more intuitive way to state the idea. Condition 4 insures the following "Archimedean" result: if $x<y$ in a physical sense in which we can compare physical sizes, then we can extend *x* by some finite number *n* of iterations so that the result *nx* is bigger than *y.* That is, Condition 4 entails the following theorem:

Theorem. Let $<B,\leq,\blacklozenge>$ be a positive extensive structure, and $x,y \in B$. If $x<y$, then there exists a positive integer *n* such that $y\leq nx$

Proof. Assume

(1) $x<y$.

By Condition 4, (1) entails for $u=v=x,$

(2) $\exists n\ (x_1\blacklozenge\ldots\blacklozenge x_{n+1} \leq y_1\blacklozenge\ldots\blacklozenge y_n\blacklozenge x)$,

By Condition 3, $\leq$ is monotonic; hence (2) entails:

(3) $\exists n\ (x_1\blacklozenge\ldots\blacklozenge x_n \leq y_1\blacklozenge\ldots\blacklozenge y_n)$.

Let *m* be the least such *n,* so that:

(4) $x_1\blacklozenge\ldots\blacklozenge x_m \leq y_1\blacklozenge\ldots\blacklozenge y_m$, and

(5) not $(x_1\blacklozenge\ldots\blacklozenge x_{\tilde{m}1} \leq y_1\blacklozenge\ldots\blacklozenge y_{\tilde{m}1})$

By Condition 1, $\leq$ is complete; hence (5) entails:

(6) $y_1\blacklozenge\ldots\blacklozenge y_{\tilde{m}1} \leq x_1\blacklozenge\ldots\blacklozenge x_{\tilde{m}1.}$

By Condition 5, the structure is positive; hence,

(7) $y \leq y_1\blacklozenge\ldots\blacklozenge y_{\tilde{m}1}$.

By Condition 1, $\leq$ is transitive; hence by (6) and (7):

(8) $y \leq x_1\blacklozenge\ldots\blacklozenge x_{\tilde{m}1}$

By definition, (8) may be rephrased:

(9) $y \leq (\tilde{m}! x.$ QED.

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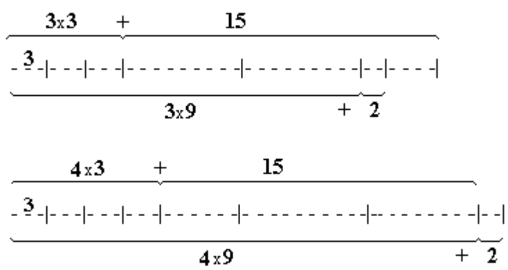
[1]Reference: David H. Krantz, R. Duncan Luce, Patrick Suppes, and Amos Taversku, *Foundations of the Theory of Measurement.* vol I. (N.Y.: Academic Press).

Example. To see that the Archimedean property as stated in Condition 4 fits measurement cases, consider the example of 3≤9. Even though 3≤9 we can increase the size of 3, say three times, and then add 6 we get 15, a number that is larger than a similar augmentation of the larger number 9 by the same factor and then adding a lesser number, say 2:

$$3x3 + 6 > 3x9 + 2$$

But eventually the fact that 9 is larger than 3 emerges if we up the increase, say in this case from 3 times to 4. We then arrive at the number predicted by the Archimedean property, namely one that makes the increase of the larger number, in this case 9, with 2 added to it , greater than the increase of the smaller number, in this case 3, with a number greater than 2, in this case 15, added to it: $3x3 + 6 \leq 4x9 + 2$



Any number *n* greater than 4 will continue to fit the Archimedean inequality. The fact that this sort of property holds depends on the numerical measurability of concatenation increments and (somewhat surprisingly given the arcaneness of its formulation) is characteristic of structures that admit of numerical measurement.

That the notion of extensive structure captures the necessary and sufficient conditions for the possibility of arithmetical measurement is shown by the following theorem:

Theorem. Let *B* be a non-empty set, ≤ a binary relation on *B* and ♦ a binary operation closed on *B*. Then <*B*,≤, ♦> is a closed extensive structure iff there exists a function *m* mapping *B* into the set of real numbers such that for all $x,y \in B$,

1. $x \leq y$ iff $m(a) \leq m(y)$,
2. $m(a \blacklozenge b) = m(a) + m(b)$.

Further, a function *m′* satisfies 1 and 2 iff there exists an *n* such that $0<n$ and for any $x \in B$, $m'(x)=nm(x)$. (That is, intuitively, any other measurement assignment *m′* will be a "scale" value of *m.*) Moreover, the structure is positive iff any $x \in B$, $0<m(x)$

PART II. NUMBERS AND THE PROPERTIES OF THE INFINITE[2]

1. The Natural Numbers

One of the core intuitions about what constitutes the infinite is formulated in terms of counting. A collection is infinite if when we start counting its elements we do not come to an end. To make sense of this we must first make sense of what it is to count, and we must do so in a way that allows for the possibility that counting will not terminate. For this purpose we invent a system of counting

[2] For background on the history of the theory of numbers as developed within logic see http://planetmath.org and Raymond L. Wilder, *Introduction to the Foundations of Mathematics*, 2nd ed. (New York: Wiley, 1967).

markers. We might for some purposes use marks on a bone like the caveman. But we want an unending collection of markers, all in order, and all different. For that purpose we construct using the tools of set theory the collection of "whole" or counting numbers: 0,1,2,3,…. Logicians call these natural numbers.

Natural numbers were first studied in modern logic means of axiom systems. Below is an example of how the theory was first formulated. A series of "primitive" or undefined terms is listed, and then a series of axioms.

Peano's Postulates (Axioms) for Arithmetic (*Arithmetices Principia*, 1889)

Primitive Symbol:	English Translation:	Mathematical Idea:
$\mathbb{N}$	*the set of natural numbers*	{0,1,2,...}
$\boldsymbol{S}(x)=y$	*the successor of x is y*	the successor relation
$\in$	*is a member of*	set membership

The Postulates

Formulation in English:	Formulation in Logical Notation:
1. 0 is a natural number.	$0\in\mathbb{N}$
2. Natural numbers are closed under successor.	$\forall x[x\in\mathbb{N}\rightarrow\boldsymbol{S}(x)\in\mathbb{N}]$
3. 0 is the successor of no natural number.	$\forall x[x\in\mathbb{N}\rightarrow\sim\boldsymbol{S}(x)=0)]$
4. If the successors of two natural numbers are the same, so are those numbers.	$\forall x\forall y([\boldsymbol{S}(x)=\boldsymbol{S}(y)]\rightarrow x=y)$
5. **Mathematical Induction**. If 0 has a property (is in A) and if a natural number has that property (is in A) only if its successor does also, then all natural numbers have that property (are in A).	$\{0\in A\&x\forall y([x\in\mathbb{N}\ \&\ y\in\mathbb{N}\ \&x\in A\&\boldsymbol{S}(x)=y]\rightarrow y\in A)\}\rightarrow\forall x(x\in\mathbb{N}\rightarrow x\in A)$

Today these same ideas are recast in a framework that defines an "abstract structure" of natural numbers.

Definition. The structure of ***natural numbers*** is the structure $<\mathbb{N},\leq,\boldsymbol{S},+,\bullet,0,1>$ such that

1. $\boldsymbol{S}$ is a unary operation (the ***successor*** operation) on sets such that $\boldsymbol{S}(x) = x\cup\{x\}$
2. $0=\varnothing$ and $1=\boldsymbol{S}(0)$
3. $\mathbb{N}$ (the set of ***natural numbers***) is the least set B such that
   a. $\varnothing\in B$
   b. for any x, if $x\in B$, then $\boldsymbol{S}(x)\in B$,
   c. nothing else is in B

4. $\leq$ is a binary relation on $\mathbb{N}$ (the ***les than*** relation) defined as follows: $x\leq y$ iff $x\subseteq y$. (By convention $x<y$ abbreviates $x\leq y$ and $x\neq y$.)

2. + is a binary operation of $\mathbb{N}$ (***addition***) defined (recursively):
   a. for all x in $\mathbb{N}$, $x + 0 = x$,
   b. for all x and y $\mathbb{N}$, $x + \boldsymbol{S}(y) = \boldsymbol{S}(x + y)$
3. $\bullet$ is a binary operation (***multiplication***) of $\mathbb{N}$ defined (recursively):
   a. for all x in $\mathbb{N}$, $x\bullet 0 = 0$,
   b. for all x and y $\mathbb{N}$, $x\bullet\boldsymbol{S}(y) = (x\bullet y) + x$

Definitions $2=\boldsymbol{S}(1)$, $3=\boldsymbol{S}(2)$, $4=\boldsymbol{S}(3)$, etc.

Note that according to these definitions each natural number is a set. The set definition is designed to insure that their set theoretic properties coincide with numerical properties that we are more familiar with.

0 is the empty set ∅. Hence 0 is the set with no members.

1 is the set containing 0, i.e. 1 is the set containing ∅: 1={0}={∅}. Hence 1 is a set with just one member. Note also that since ∅ is a subset of every set, ∅ is a subset of {∅}.

2 is the set containing 0 and 1, i.e. 2={0,1}={∅,{∅}}. Hence 2 is a set containing just two members, and is in fact the set containing all the natural numbers less than itself. Note also that both ∅, which is 0, and {∅}, which is 1, are subsets of {∅,{∅}}, which is 2. Hence the relation ⊆ of *subset* captures the *less than* relation ≤ for numbers less than 2.

3 is the set containing 0,1, and 2, i.e. 3={0,1,2}={∅,{∅},{∅,{∅}}}. Hence 3 is a set that contains just three elements, namely all the natural numbers less than 3. Note also that ∅, which is 0, and {∅}, which is 1, and {∅,{∅}}, which is 2, are subsets of {∅,{∅},{∅,{∅}}}, which is 3. Hence the relation ⊆ of *subset* captures the *less than* relation ≤ for numbers less than 3.

In general, the definitions insure that a natural number *n* is a set that contains exactly *n* elements, and that these are exactly all the natural numbers less than *n*. Moreover, the numbers are defined as sets in such a way that any number less than *n* is a subset of *n*.

Theorem

1. ***S*** is defined for all elements of ℕ ,and ***S***(x)≠0
2. for all *x* and *y* of ℕ , if ***S***(x)=***S***(y) then x=y
3. (**Mathematical Induction**) for any set A, if 0∈A and, for any *x,* if x∈A then ***S***(x)∈A, then for any x∈ℕ , x∈A)
4. ≤ is a total ordering of ℕ ,
5. < ℕ ,≤,=,+,•,0,1> where = is the identity relation on sets in ℕ is an ordered ring with identity relation =.

It is a relatively easy matter to prove that ℕ is infinite in the sense of Cantor. Though we shall not prove so here ℕ is also the smallest infinite set (any set that is infinite is either larger than or equipollent to it).

Theorems.

1. ℕ is infinite.
2. For any *A*, if *A* is infinite, then either A≈ℕ or ℕ⩽*A.* (See Part III below for definitions.)

Definition. A set *A* is ***countably infinite*** or ***denumerable*** iff it can be put into 1-1 correspondence with ℕ , i.e. A≈ℕ

2. The Integers

Infinite however goes not only upward but downward. Indeed the discovery that an infinite descending series is logically possible was one of the advances in mathematics with direct relevance to issues in philosophy, like whether infinitely receding causal chains are logically coherent. The next step in the definition of number is, therefore, to introduce the negative integers.

Definition. The structure of ***integers*** is the structure $<\mathbb{Z}, \leq,+,\bullet,-,0,1>$ such that

1. $\mathbb{Z}$ is the set of (***integers***) defined as the set of all equivalence classes $[n,n']_{\equiv}$ of ordered pairs of natural numbers determined by the equivalence relation $\equiv$ defined (on pairs of natural numbers) as follows: for any x,y,u and v in $\mathbb{N}$,
   $<x,y> \equiv <u,v>$ iff $x+v = y+u$.
   That is, for any x and y in $\mathbb{N}$,
   $[x,y]_{\equiv} = \{<u,v> | u\in\mathbb{N} \& v\in \mathbb{N} \& x+v = y+u\}$
2. $\leq$ is a binary (***less than***) relation on $\mathbb{Z}$ defined as follows: $[x,y]_{\equiv} \leq [u,v]_{\equiv}$ iff $x+v \leq y+u$ in $\mathbb{N}$. (By convention $x<y$ abbreviates $x\leq y$ and $x\neq y$.)
3. $+$ is a binary (***addition***) operation on $\mathbb{Z}$ defined as follows: $[x,y]_{\equiv} + [u,v]_{\equiv} = [x+u, y+v]_{\equiv}$
4. $\bullet$ is the binary (***multiplication***) operation on $\mathbb{Z}$ defined as follows: $[x,y]_{\equiv} \bullet [u,v]_{\equiv} = [(x\bullet u)+(y\bullet v), (x\bullet v)+(y\bullet u)]_{\equiv}$
5. $0 = [0,0]_{\equiv}$ and $1=[1,0]_{\equiv}=[2,1]=...$, $-1=[0,1]=[1,2]=...$
6. $-$ is a unary (***minus***) operation on $\mathbb{Z}$ defined as follows: $-[x,y]_{\equiv} = [y,x]_{\equiv}$

Definitions. In the structure of integers $<\mathbb{Z}, \leq,+,\bullet,-,0,1>$

$n =_{def} [x,y]_{\equiv}$ such that $y\leq x$, and $x = y + n$,
$-n =_{def} [x,y]_{\equiv}$ such that $x\leq y$, and $x + n = y$
$|n| = n$ if $n = [x,y]_{\equiv}$ such that $y\leq x$, and $|-n| = n$ if $-n = [x,y]_{\equiv}$ such that $x\leq y$.

Theorem. $<\mathbb{Z}, \leq,=,+,\bullet,-,0,1>$ such that such that $<\mathbb{Z}, \leq,+,\bullet,-,0,1>$ is the structure of integers and $=$ is the identity relation on sets is an ordered field.

Theorem. If $<\mathbb{Z}, \leq,+,\bullet,-,0,1>$is the structure of integers, then $<\mathbb{Z}, \leq>$ is a total but not a dense ordering.

3. Rational Numbers

Integers however do not exhibit an important feature of the infinite as we encounter in cases of divisibility. Unlike distances and times, which always seem to admit a degree intermediate between a larger and smaller, integers are not "dense". For that property we must add to the integers all whole number fractions, thus generating the rational numbers.

Definition. The structure of ***rational numbers*** is the structure $<\mathbb{Q}, \leq,+,\bullet,-,0,1>$ such that

1. $\mathbb{Q}$ is the set of (***rationals***) defined as the set of all equivalence classes $[n,n']_{\equiv}$ of ordered pairs of integers, such that $n' \neq 0$, determined by the equivalence relation $\equiv$ defined (on pairs of rationals) as follows: for any x,y,u and v in$\mathbb{Z}$,
   $<x,y> \equiv <u,v>$ iff $x\bullet v = y\bullet u$.
   That is, for any x and y in $\mathbb{Z}$ such that $y \neq 0$,
   $[x,y]_{\equiv} = \{<u,v> | u\in\mathbb{Z} \& v\in \mathbb{Z} \& x\bullet v = y\bullet u\}$
   It is conventional to abbreviate $[x,y]_{\equiv}$ in "fraction" notation as x/y or $\dfrac{x}{y}$
2. $\leq$ is a binary (***less than***) relation on $\mathbb{Q}$ defined as follows: $\dfrac{x}{y} \leq \dfrac{u}{v}$ iff $x\bullet v \leq y\bullet u$ in $\mathbb{Z}$
   (By convention $x<y$ abbreviates $x\leq y$ and $x\neq y$.)
3. $+$ is a binary (***addition***) operation on $\mathbb{Q}$ defined as follows:
   $$\frac{x}{y} + \frac{u}{v} = \frac{(x\bullet v)+(y\bullet u)}{y\bullet v}$$

4. $\bullet$ is the binary (***multiplication***) operation on $\mathbb{Q}$ defined as follows:
$$\frac{x}{y} \bullet \frac{u}{v} = \frac{x \bullet u}{y \bullet v}$$
5. $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$
6. $-$ is a unary (***minus***) operation on $\mathbb{Q}$ defined as follows: $-\frac{x}{y} = \frac{-x}{y}$

Theorem. $<\mathbb{Q}, \leq,=,+,\bullet,-,0,1>$ such that $<\mathbb{Q}, \leq,+,\bullet,-,0,1>$ is the structure of rationals and = is the identity relation on sets is an ordered field.

Theorem. If $<\mathbb{Q}, \leq,+,\bullet,-,0,1>$ is the structure of rationals, then $<\mathbb{Q}, \leq>$ is a dense but not a continuous ordering.

4. Real Numbers

In remains to construct the real numbers which culminate our structural progression by embodying continuity.

Definition. The structure of ***real numbers*** is the structure $<\mathbb{R}, \leq,|\ |,+,\bullet,-,0,1,^{-1}>$ such that
1. $\mathbb{R}$ is the set of (***reals***) defined as the set of all subsets C of $\mathbb{Q}$ such that
   a. C and $-C$ are non-empty
   b. C and $-C$ partition $\mathbb{Q}$ (i.e. $C \cup -C = \mathbb{Q}$)
   c. C contains no greatest element (i.e. for any x in C, there is a y in C such that $x \leq y$)
   d. C is closed dowardly under $\leq$ (i.e. for any x in C and any y in $\mathbb{Q}$, if $y \leq x$, then y is in C) (Note that the pair $<C, -C>$ meeting conditions a-d is called a ***Dedekind cut*** and Dedekind himself identified reals with these pairs.)
2. $\leq$ is a binary (***less than***) relation on $\mathbb{R}$ defined as follows: $x \leq y$ iff $x \subseteq y$. (By convention $x<y$ abbreviates $x \leq y$ and $x \neq y$.)
3. $-$ is a unary (***minus***) operation on the reals defined as follows: $-x = \{y \mid y \in \mathbb{Q} \ \& -y \notin x \ \& -y$ is not the least element of $\mathbb{Q} - x\}$
4. + is a binary (***addition***) operation on $\mathbb{R}$ defined as follows:
$$x + y = \{u+v \mid u \in \mathbb{Q} \ \& \ v \in \mathbb{Q} \ \& \ u \in x \ \& \ v \in y\}$$
5. 0 (the ***additive identity***) is $\{x \mid x \in \mathbb{Q}$ and $x \leq 0\}$
6. 1 (the ***multiplicative identity***) is $\{x \mid x \in \mathbb{Q}$ and $x \leq 1\}$
7. $|\ |$ is the binary (***absolute value***) operation on $\mathbb{R}$ defined for x in $\mathbb{R}$,
   if $0 \leq x$, $|x| = x$, and
   if $x \leq 0$, $|x| = -x$
8. $\bullet$ is the binary (***multiplication***) operation on $\mathbb{R}$ defined in stages as follows:
   a. $\bullet$ is first defined for reals greater than 0: if $0<x$ and $0<y$, then
      $x \bullet y = \{z \mid z \in \mathbb{Q}$ &
      (either $z \leq 0$ or (there exist u and v in $\mathbb{Q}$ such that $0<u$ & $0<v$ and $z = u \bullet v\}$
   b. $\bullet$ is now defined for all reals: for any x and y in $\mathbb{R}$,
      if $x=0$ or $y=0$, then $x \bullet y = 0$;
      if ($0<x$ & $0<y$) or ($x<0$ & $y<0$), then $x \bullet y = |x| \bullet y|$;
      if ($0<x$ & $y<0$) or ($0<x$ & $y<0$), then $x \bullet y = -(|x| \bullet y|)$.
9. $^{-1}$ is the unary (***inverse***) operation on $\mathbb{R}$ defined
   if $0 \leq x$, $x^{-1} = \{y \mid y \in \mathbb{Q}$ and (either $y \leq 0$ or [$0<y$ and $\frac{1}{y} \notin x$ and $\frac{1}{y}$ is not the least element of $\mathbb{Q} - x$])}

if $x \leq 0$, $x^{-1} = -(|x|)^{-1}$

Definitions. In the structure of reals $<\mathbb{R}, \leq, |\ |, +, \bullet, -, 0, 1, ^{-1}>$, n in $\mathbb{R}$ is said to be ***irrational*** iff n contains no least element.

It is now possible to prove what the special properties related to infinity that the real numbers exemplify. Though like the rationals, the real line is dense – between any two "points" on "the line," there is a third -- the real line has the property which the rationals lack that it may be divided at a point is such a way that other reals may approach is closer and closer *ad infinitum* both from below and above.

Theorem. Let $<\mathbb{R}, \leq, +, \bullet, -, 0, 1>$ be the structure of the real numbers

1. (**The Completeness Property.)** $<\mathbb{R}, \leq>$ is continuous and has the Dedekind completeness property, i.e. every non-empty subset A of real numbers that is bounded (i.e. such that there is some x in $\mathbb{R}$ such that for any $y \in A$, $y \leq x$) has a $\leq$-least upper bound (a supremum) in $\mathbb{R}$.

Proof. Let A be an arbitrary non-empty subset of $\mathbb{R}$ and let x in $\mathbb{R}$ be an arbitrary upper bound of A, i.e. let us assume that for any $y \in A$, $y \leq x$. We define a set , namely the union of all the reals in A, which are themselves sets of rationals: $n = \bigcup\{z | z \in A\}$. We claim that (1) n is a real, (2) n is an upper bound of all elements in A and (3) n is a least upper bound of elements in A.

To show (1) we must show that n meets the defining conditions a-d of $\mathbb{R}$. (a) By definition the elements of A are in $\mathbb{R}$ and are therefore themselves non-empty. Hence their union is non-empty. Moreover. since x is a real number, it is non-empty. Hence $\mathbb{R}-n$, which contains x, is non-empty. (b) Moreover by set theory n and $\mathbb{R}-n$ are disjoint and their union is $\mathbb{R}$. They therefore partition $\mathbb{R}$. (c) Let us assume for a reduction to the absurd that for some x in n, for every y in n, $y<x$. Then, since x is in n then for some z in A, $x \in z$. But then x would be the greater than any element in z, and z would both be a real number and contain a greatest element, which is absurd. Hence the assumption leads to a contradiction and is false. Hence that for all x in n, for some y in n, $x \leq y$. (d) Let x be an arbitrary element in n and y an arbitrary rational number in $\mathbb{Q}$. Suppose further that $y \leq x$. Since x is in n, x is in some real, call it z, in A. Now since z is a real it is downwardly closed. Hence since $x \in z$ and $y \in \mathbb{Q}$, $y \in z$. But $z \subseteq n$. Hence $y \in n$.

(2) Let x be an arbitrary element of A. Then $x \subseteq \bigcup\{z | z \in A\}$, i.e. $x \subseteq n$. But then by definition $x \leq n$. Hence for all $x \in A$, $x \leq n$. Hence n is an $\leq$-upper bound of A.

(3) Since x is an arbitrary upper bound of A, it will suffice to show $n \leq x$. Let $y \in n$. Then, for some z in A¸ $y \in z$. But since x is an upper bound of all elements of A, it is an upper bound of z. Hence, since $y \in z$, $y \leq x$. Hence n is a least upper bound of A. **QED.**

2. If B and $-B$ are non-empty subsets of $\mathbb{R}$ that partition $\mathbb{R}$ into upper and lower "halves" (i.e. $B \cup -B = \mathbb{R}$, and for any x if $x \in B$ then for any y if $y \in -B$, then $x \leq y$) then there exist a unique z in $\mathbb{R}$ such that z is the $\leq$-greatest element of B or the $\leq$-least element of $-B$:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. (

Though it is a relatively simple matter to show directly that natural numbers, integers, and rationals all exhibit Archimedean properties, that fact that reals are Archimedean (and hence that the more basic types are too) requires the completeness property.

Corollaries (Archimedean Properties)

1. (**Fundamental Archimedean Property** of $\mathbb{R}$) For any x, if $x\in\mathbb{R}$, then there exists a y such that $y\in\mathbb{N}$ and $x<y$.

Proof. Let x be an arbitrary real number in $\mathbb{R}$ and let A be $\{y|y\in\mathbb{N}\ \&\ y\leq x\}$. Case 1. A is the empty set. Now $1\notin\varnothing$. Hence $1\notin\{y|y\in\mathbb{N}\ \&\ y\leq x\}$. Hence not($1\in\{y|y\in\mathbb{N}\ \&\ y\leq x\}$). Hence not($1\in\mathbb{N}\ \&\ 1\leq x$). Thus either $1\notin\mathbb{N}$ or (not$1\leq x$). But $1\in\mathbb{N}$. Hence not $1\leq x$. Hence $x<1$. Case 2. A is non-empty. Since $\mathbb{N}\subseteq\mathbb{R}$ and $A\subseteq\mathbb{N}$, $A\subseteq\mathbb{R}$. Moreover, by definition every element z of A is such that $z\leq x$. Hence x is an upper bound of A. Hence A is a bounded non-empty subset of $\mathbb{R}$. Therefore, by the completeness property, there is some w in $\mathbb{R}$ that is a least upper bound of A. Moreover $w-1<w$ and is therefore not a least upper bound of A. Hence there exist a u in A such that $w-1<u$. Let y be $u+1$. Since by definition of A, $u\in\mathbb{N}$, $y\in\mathbb{N}$. Moreover, since $w<y$ and w is an upper bound of A, $y\notin A$. Further, since $y\notin A$ by definition of A, not($y\in\mathbb{N}\ \&\ y\leq x$), i.e. either $y\notin\mathbb{N}$ or not $y\leq x$. But $y\in\mathbb{N}$. Hence, not $y\leq x$. Therefore $x<y$. **QED.**

2. For any x and y, if x and y are in $\mathbb{R}$ such that $0<x$, there exists a z such that $z\in\mathbb{N}$ and $y<z\bullet x$.

Proof. Let x and y be arbitrary real numbers in $\mathbb{R}$ such that $0<x$. Since x and y are in $\mathbb{R}$ and $x\neq 0$, $\frac{y}{x}$ is defined and is in $\mathbb{R}$.By the previous theorem there is some $z\in\mathbb{N}$ such that $\frac{y}{x}<z$. It follows then that $y<x\bullet z$. **QED.**

3. For any x and y, if $x\in\mathbb{R}$ and $0<x$, then there exists a y such that $y\in\mathbb{N}$ and

$$0<\frac{1}{y}<x.$$

5. Cauchy Sequences and Metric Spaces

An equally well known and theoretically somewhat more fruitful definition of the reals in terms of the rational makes use of what are called ***Cauchy sequences*** of rationals after their discoverer. Like the sets making up a Dedekind cut these approach ever closer to real in a way characteristic of it. The basic definitions are given below both because the construction is well known as a classical alternative to Dedekind's, and because we will later indicate how thee definitions provide the grounds for showing that the properties of completeness represented by the real numbers may be studied in yet more abstract structures called metric spaces.

Definitions

1. If s and s' are sequences on A, then the ***interleave sequence of*** s ***and*** s' in A is the sequence $s''=s_1, s'_1, s_2, s'_2, s_3, s'_3, \ldots, s_n, s'_n,\ldots$, i.e.
   i. $s''(x)=s\left(\frac{x}{2}\right)$ if x is even,
   ii. $s''(x)=s'\left(\frac{x}{2}\right)$ if x is odd.

2. If s is a sequence in the set $\mathbb{Q}$ of rational numbers, then s is a ***Cauchy sequence of rationals*** iff for every rational number ε in $\mathbb{Q}$, if $0<\varepsilon$, there exist a natural number x in $\mathbb{N}$ such that for any y and z in $\mathbb{Q}$, if $x<y$ and $x<z$, then $|s_y-s_z|<\varepsilon$.
3. If s and s' are Cauchy sequences of rationals, then $s\equiv s'$ iff the interleave sequence s'' of s and s' in sequence in $\mathbb{Q}$ of rational numbers, is also a Cauchy sequence of rationals.
4. If s is a Cauchy sequence in the set $\mathbb{Q}$ of rational numbers, then the $\equiv$-equivalence class of s, briefly $[s]_\equiv$, is $\{s' \mid s'$ is a Cauchy sequence of rationals and $s\equiv s'\}$.

Theorem. The set $\mathbb{R}$ of real numbers is the set of all equivalence classes $[s]_\equiv$ such that s is a Cauchy sequence of rationals.

Though the definitions of the various types of numbers and their properties are now modern classics, modern mathematics has shown that they are not the most fundamental or abstract definitions possible for structures that exhibit the sort of continuity typified by the real number line. The notion of "completeness" is definable more abstractly for what are called of metric spaces. The basic definitions are given here to indicate the direction such studies take.

Definitions

1. A ***sequence*** in A is a function s from the set of natural numbers $\mathbb{N}$ into some set A. When the context makes it unambiguous that it is the sequence s that is in question, the series of s-values $s(1)$, $s(2)$, $s(3)$, …, $s(n)$, … for the arguments $1,2,3,..,n,…$ is customarily written as $s_1, s_2, s_3, …, s_n,…$.
2. A ***metric space*** is a structure $<S,d>$ such that d is a function (called the ***metric*** or ***distance function***) on pairs of real numbers (i.e. the domain of d is $\mathbb{R}\times\mathbb{R}$) such that for any x, y, and z in S,
   a. if $x=y$ then $d(x,y)=0$;
   b. if $x\neq y$ then $0< d(x,y)$;
   c. $d(x,y) = d(y,x)$;
   d. $d(x,z) \leq d(x,y) + d(y,z)$.
3. If $<S,d>$ is a metric space and s is a sequence in S, then s is a ***Cauchy sequence*** in $<S,d>$ iff for every real number ε if $0<\varepsilon$, there exist a natural number x in $\mathbb{N}$ such that for any y and z in , if $x<y$ and $x<z$, then $d(s_y, s_z)<\varepsilon$.
4. If $<S,d>$ is a metric space, x is an element of S, and s is a sequence in S, then s ***converges to*** x and x is ***the limit point*** of s iff for every real number ε if $0<\varepsilon$, there exist a natural number y in $\mathbb{N}$ such that for all z in $\mathbb{N}$, if $y<z$, $d(x, s_z)<\varepsilon$. If there is an x in S such that s converges to x then s is said to ***converge***.
5. A metric space $<S,d>$ is ***complete*** if every Cauchy sequence in S converges.

Theorem. If $<\mathbb{R}, \leq, |\,|, +, \bullet, -, 0, 1, ^{-1}>$ be the structure of the real numbers, then $<\mathbb{R}, |\,|>$ is a complete metric space.

6. Hyperreals, Introduction

Those ancient, mediaeval and Enlightenment philosophers and mathematicians discussed seriously and made efforts to construct coherent accounts of true "infinitesimals" the idea was not needed to explain modern mathematics grew up with the invention of the infinitesimal calculus by Newton and Leibniz in the seventieth century. The theory of real numbers, which rejects the notion of a true infinitesimal, has proved adequate to account for the calculus and number theory.

In the `940 however Abraham Robinson invented a theory of numbers that supplements the reals by adding to them new numbers that fall in between and have the property of hovering as "clouds" infinitesimally close – in a well defined sense – to both 0 and to each real.

Like the construction of the earlier numerical structures that of the hyperreals is defined in terms to the next simpler structure, in this case the reals. Just as integers are a certain equivalent class of naturals and rational of integers, and reals of rationals, the hyperreals are special equivalence classes of defined in terms of reals. In particular, they are equivalence classes of infinite sequences of reals of a certain sort. Now, an infinite sequence of reals, indicated by the notation s_1, s_2, s_3, …, s_n,…, or even more briefly as $\{s_n\}$, is a function s from the set of natural numbers $\mathbb{N}$ into that of the reals $\mathbb{R}$. The sequence is said to "index" elements of $\mathbb{R}$ because for each "address" in $\mathbb{N}$ it assigns an "occupant in $\mathbb{R}$. The construction identifies hyper reals with equivalent classes of such sequences. The ordinary real number r then becomes identifies with the equivalence class of the special series that just repeats r and infinite number of times: $r = r_1, r_2, r_3, \ldots, r_n,\ldots$. The "new" numbers are those formed by series that do not have this constant repletion.

The trick is to define the equivalence relation $\equiv$ on sets of these series so that the right results follow from the definition. For example, we want the resulting "numbers" to form an ordered field, as well as allow for the relevant definition of "infinitesimal". The construction proceeds by first defining an order relation $\leq$ on sets of infinite sequences of reals, and then defining $\equiv$ in terms of $\leq$.

How should $\leq$ be defined? . We might simply say that one series was "less than" another if each number at each address was so. That is, we might say $s_1, s_2, s_3, \ldots, s_n,\ldots \leq s'_1, s'_2, s'_3, \ldots, s'_n,\ldots$ iff for each n , $s_n,\leq s'_n$. But this is too restrictive. We are going to allow one series to be "less than" another if just a selected subsets of their components are so. The trick is to pick out a way to characterize the relevant subsets of "addresses" to be considered.

The construction first sets out the relevant subsets of $\mathbb{N}$ to be used in these order comparisons. It turns out that an adequate family of subset for defining the right notion of $\leq$ are those that we arrayed vertically in a diagram in terms of the subset relation $\subseteq$ make a picture that looks something like a filter. It

We pause to define this family precisely. We first repeat the earlier definition of the algebra of subsets of a given set *B.*

Definition. A structure $<B,\wedge,\vee,-,0,1>$ is a ***Boolean algebra*** iff it is a structure satisfying the following conditions. Let *x, y* and *z* be arbitrary members of *B*.

1. $<B,\wedge,\vee>$ is a lattice, i.e.
   - L1. $x\wedge y=y\wedge x$; $x\vee y=y\vee x$
   - L2. $(x\wedge y)\wedge z=x\wedge(y\wedge z)$; $(x\vee y)\vee z=x\vee(y\vee z)$;
   - L3. $x\wedge x=x=x\vee x$;
   - L4. $x\vee(x\wedge y)=x=x\wedge(x\vee y)$.
2. $<B,\leq>$ is a partially ordered structure, i.e. by definition $x\leq y \Leftrightarrow x\wedge y=x \Leftrightarrow x\vee y=y$ and
   - P1. $x\leq x$;

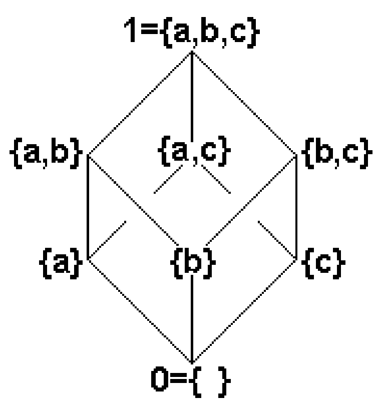
P2 $x \leq y$ & $y \leq z$.⇒ $x \leq z$;
P3. $x \leq y$ & $y \leq x$.⇒ $x = y$.

3. $<B, \wedge, \vee>$ is distributive, i.e.
   D1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
   D2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
4. 0 and 1 are respectively the least and greatest element of B in $<B, \wedge, \vee, 0, 1>$, i.e.
   G1. $0 \leq x \leq 1$;
   G2. $1 \wedge x = x$;
   G3. $1 \vee x = 1$;
   G4. $0 \wedge x = 0$;
   G5. $0 \vee x = x$.
5. $-$ is a unique complementation operation on one-place operation on $<B, \wedge, \vee, -, 0, 1>$, i.e. B is closed under $-$ and
   C1. $x \wedge -x = 0$
   C2. $x \vee -x = 1$
   C3. $--x = x$, $-1 = 0$, $-0 = 1$;
   C4. $x \leq y \Leftrightarrow x \wedge -y = 0 \Leftrightarrow -y \leq -x \Leftrightarrow -x \vee y = 1$
   C5. $-(x \wedge y) = -x \vee -y$, $-(x \vee y) = -x \wedge -y$.

Theorem. $<B, \wedge, \vee, -, 0, 1>$ is a Boolean algebra iff $\wedge$ and $\vee$ are binary and $-$ a unary operation on B under which B is closed, $1, 0 \in B$ and

L1. $x \wedge y = y \wedge x$; $x \vee y = y \vee x$;
D1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
D2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
C1. $x \wedge -x = 1$
C2. $x \vee -x = 0$
G2. $1 \wedge x = x$;
G5. $0 \vee x = x$;

Example. A three element Boolean Algebra

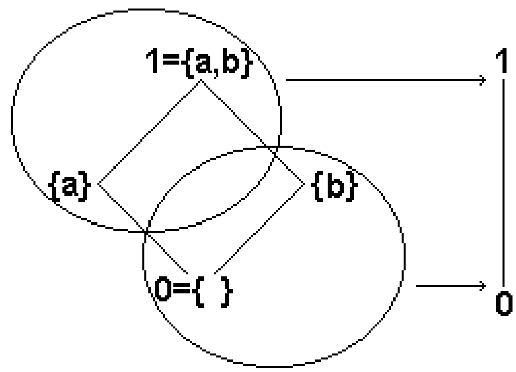


A Boolean Algebra of the Power set of {a,b,c}

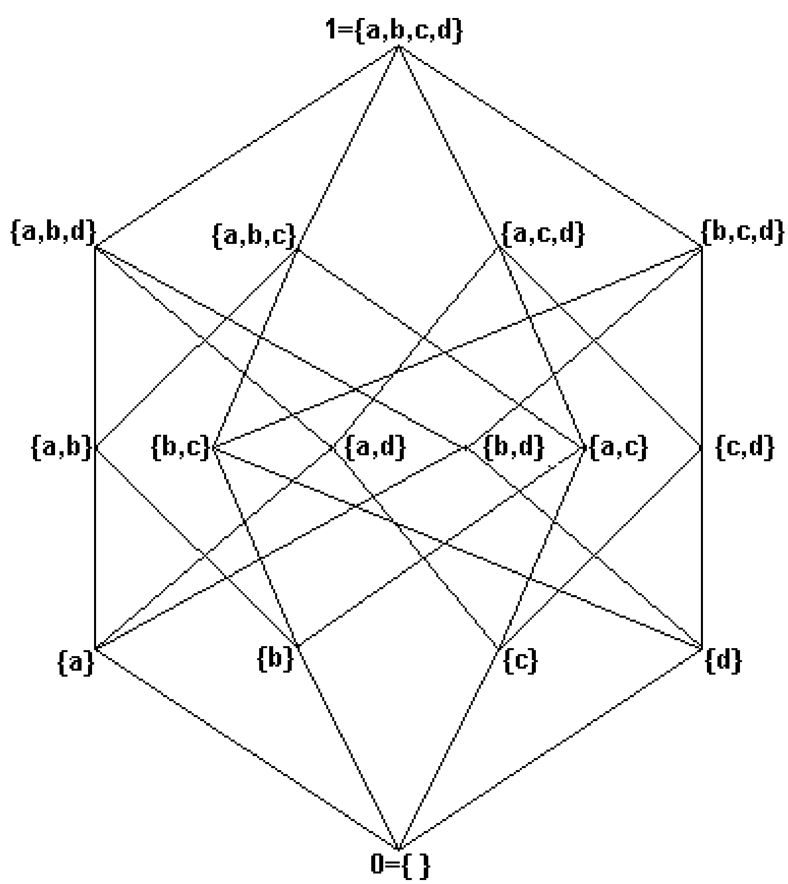
We shall let $\mathcal{B} = <B, \wedge, \vee, -, 0, 1>$ range over Boolean algebras, distinguish one algebra from another by prime marks on its various components.

Theorem. Although any congruence relation for a Boolean Algebra $<B, \wedge, \vee, -, 0, 1>$ has (by definition) the substitution property for $\wedge, \vee, -$, it does not in general have the substitution property for $\leq$. That is, there are some Boolean Algebras with congruence relation $\equiv$ such that for some a,b,c in B, a≡b, c≡d, and a≤c, yet not(b≤c).

Consider the function ϕ diagrammed below mapping one Boolean algebra to another and hence determining a congruence relation $\equiv_\phi$. That is ϕ is defined: Here $\phi(1)=1$ $\phi(a)=1$, $\phi(b)=0$, $\phi(0)=0$.) Here $\phi(x\wedge y)=\phi(x)\wedge\phi(y)$ and $x\equiv_\phi y$ & $z\equiv_\phi w$.$\Rightarrow$ $x\wedge z\equiv_\phi y\wedge w$, and likewise for $\vee$. But, $0\equiv_\phi b$ & $1\equiv_\phi a$ & $0\leq 1$, yet not($b\leq 0$). **QED**



Example. A four element Boolean Algebra



A Boolean Algebra of the Power set of {a,b,c,d}

7. Filters and Ideals

A important subset of the universe of a Boolean algebra is the set of elements above *x*, or dually the elements below *x.* The former is called a *filter*, the latter an *ideal.* A maximal filter of *x* and dual maximal ideal of $-x$ have the very nice property that they partition the algebra into just two equivalence classes that also determine a congruence relation. In other words, they proved a two element Boolean algebra with the "same structure" as the original. This binary structure "represents" the original and allows all Boolean algebras to be simplified into the structure on {0,1}. In the next section we shall apply this representation to the matrix interpretations of classical logic, where we shall find that the family of Boolean algebras is characteristic of classical deducibility, but by means of the representation theorem these may all be simplified to the familiar classical matrix on {T,F}.

Definitions. Filters and Ideals.[3]

Let $\mathfrak{B}=\langle B,\wedge,\vee,-,0,1\rangle$ be a Boolean algebra and $A\subseteq B$.

A is a ***filter*** on $\mathfrak{B}$ iff

1. $\forall x,y\in B$, $x\in A\Rightarrow x\vee y\in A$, and
2. $\forall x,y\in B$, $x,y\in A\Rightarrow x\wedge y\in A$

(equivalently, iff $\forall x,y\in B$, $x,y\in A\Leftrightarrow x\wedge y\in A$).

A is an ***ideal*** on $\mathfrak{B}$ iff

1. $\forall x,y\in B$, $x\in A\Rightarrow x\wedge y\in A$, and
2. $\forall x,y\in B$, $x,y\in A\Rightarrow x\vee y\in A$

(equivalently, iff $\forall x,y\in B$, $x,y\in A\Leftrightarrow x\vee y\in A$).

For any $x\in B$, by $\mathbf{[x]\!\uparrow}$ we mean $\{y|x\leq y\}$ and by $\mathbf{[x]\!\downarrow}$ we mean $\{y|y\leq x\}$

Theorem. For any Boolean algebra $\mathfrak{B}=\langle B,\wedge,\vee,-,0,1\rangle$a and any $x\in B$,

$[x]\!\uparrow$ is a filter on $\mathfrak{B}$, and
$[x]\!\downarrow$ is an ideal on $\mathfrak{B}$.

Definition 3. For any Boolean algebra $\mathfrak{B}=\langle B,\wedge,\vee,-,0,1\rangle$a and any $x\in B$,

$[x]\!\uparrow$ is ***the prime*** (or ***principle***) ***filter on*** $\mathfrak{B}$ ***relative to*** *x* and
$[x]\!\downarrow$ is ***the prime*** (or ***principle***) ***ideal on*** $\mathfrak{B}$ ***relative to*** *x*.

Example. The prime filter of a and the prime ideal of its complement $-$a={b,c}.

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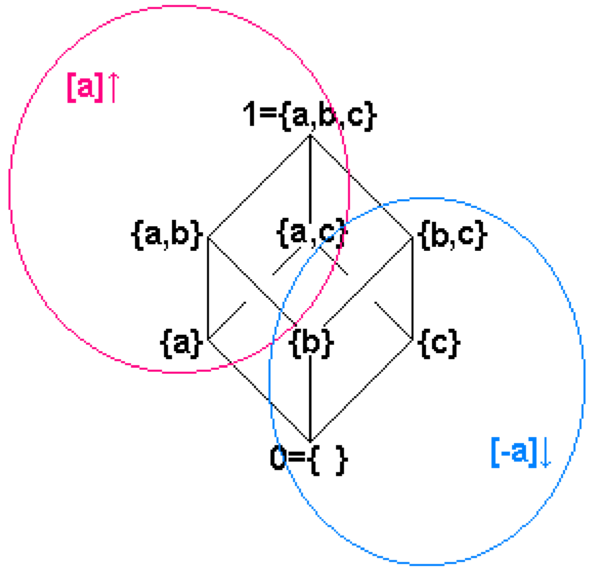
[3] A note on symbolism. We abbreviate the conjunction $x\in$A & $y\in$A as $x,y\in$A.



Theorem. For any Boolean algebra $\mathfrak{B}$=<B,∧,∨,–,0,1>, every filter/ideal of $\mathfrak{B}$ is prime iff B is finite.

Definition. A filter/ideal of a Boolean algebra $\mathfrak{B}$ is ***maximal*** iff

1. for some filter/ideal H, $B \subset H$, and
2. for any filter/ideal G, if there is a filter/ideal H such that $G \subset H$, then, if $B \subseteq G$, B=G (i.e. if G is a proper filter/ideal then B is not properly contained in it.)

Theorem. For any Boolean algebra $\mathfrak{B}$=<B,∧,∨,–,0,1>,

1. F is a maximal filter/ideal of $\mathfrak{B}$ iff, d $\forall x \in F$, not($x \in F \Leftrightarrow -x \in F$).
2. F is a maximal filter/ideal of $\mathfrak{B}$ iff, B–F is a maximal ideal/filter of $\mathfrak{B}$
3. F is a maximal ideal of $\mathfrak{B}$ iff, the function θ from B into its power set $\mathfrak{P}(B)$ defined as follows: $\forall x \in B$,

   $\theta(x)=F$ if $x \in F$, and
   $\theta(x)=B-F$ if $x \notin F$

   is a homomorphism from $\mathfrak{B}$ onto the Boolean

   <{F,B–F},∩,∪,–,F,B–F>

Definition. Let <X,≤> be a partially ordered structure.

A ***chain*** in <X,≤> is any non-empty subset Y if X such that if $x,y \in Y$ then $x \leq y$ or $y \leq x$.

An ***upper bound*** of a chain Y is <X≤> is a member x of X such that for all $y \in Y$, $y \leq x$.

An element x of is a ***maximal element*** of <X,≤> iff, for $x,y \in X$, $x \leq y \Rightarrow x=y$

Any $\subseteq$-chain of subsets A_1,A_2,… clearly posses an upper bound and lower bounds, namely their union and intersection. Since filters are closed with respect to the union of any chain of its elements, and ideal with respect to their intersection, they each contain respectively at least one upper or lower bound. Zorn's lemma insures that there is a lowest or highest respectively.

Theorem. *For any Boolean algebra $\mathfrak{B}$=<B,∧,∨,–,0,1>, any $x \in B$ and any filter/ideal F of $\mathfrak{B}$ that does not contain x, there exists a minimal/maximal (***ultra***) filter/ideal M of $\mathfrak{B}$ such that $M \subseteq F$/ $F \subseteq M$ and $x \notin F$.

Theorem. *For any Boolean algebra $\mathfrak{B}$=<B,$\wedge$,$\vee$,$-$,0,1>, and any x and y of B, if not($y \leq x$), then there exists a minimal/maximal (ultra) filter/ideal F of $\mathfrak{B}$ such that $x \in F$ and $y \notin F$.

In the case of filters, the existence of an upper bound to $\subseteq$-chains and of an ultrafilter remains is true even if the filter is limited to infinite sets. It is such ultra filters of infinite sets, called ***non-principal***, that are used to categorize the acceptable subsets of natural numbers that are used to as indices for the infinite series used below to construct the hyperreals.

8. Properties of Hyperreal[4]

The ***hyperreal numbers*** or ***nonstandard reals*** (usually denoted as $^*\mathbb{R}$) are an extension of the real numbers $\mathbb{R}$ that adds infinitely large as well as infinitesimal numbers to $\mathbb{R}$. The study of these numbers, their functions and properties is called nonstandard analysis which some find more intuitive than standard real analysis. When Isaac Newton and Gottfried Leibniz introduced differentials, they used infinitesimals and these were still regarded as useful by Leonhard Euler and Augustin Louis Cauchy. Nonetheless these concepts were from the beginning seen as suspect, notably by Bishop Berkeley, and when in the 1800s calculus was put on a firm footing through the development of the epsilon-delta definition of a limit by Augustin Louis Cauchy, Abraham Robinson showed how infinitely large and infinitesimal numbers can be rigorously defined and used to develop the field of nonstandard analysis. Because his theory in its full-fledged form makes unrestricted use of classical logic and set theory and, in particular, of the Axiom of Choice, it is suspected to be nonconstructive from the outset. The construction given below is a simplified version of Robinson's more general construction and is due to Lindstrom.

The hyperreals $^*\mathbb{R}$ form an ordered field containing the reals $\mathbb{R}$ as a subfield. Unlike the reals, the hyperreals do not form a metric space, but by virtue of their order they carry an order topology.

The hyperreals are defined in such a way that every true first-order logic statement that uses basic arithmetic (the natural numbers, plus, times, comparison) and quantifies only over the real numbers is also true if we presume that it quantifies over hyperreal numbers. For example, we can state that for every real number there is another number greater than it:

$$\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x < y$$

The same will then also hold for hyperreals:

$$\forall x \in {}^*\mathbb{R} : \exists y \in {}^*\mathbb{R} : x < y$$

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[4] Adapted from Wikipedia at http://www.campusprogram.com/reference/en/wikipedia/h/hy/hyperreal_number.html.

See also http://mathforum.org/dr.math/faq/analysis_hyperreals.html. and Jordi Gutierrez Hermoso, *Nonstandard Analysis and the Hyperreals*.

Another example is the statement that if you add 1 to a number you get a bigger number:

$\forall\, x \in \mathbb{R} : x < x + 1$

which will also hold for hyperreals:

$\forall\, x \in {}^*\mathbb{R} : x < x + 1$

This however doesn't mean that $\mathbb{R}$ and ${}^*\mathbb{R}$ behave the same. For instance, in ${}^*\mathbb{R}$ there exists an element w such that

$1 < \ldots\ 1+1+1+1 < w$

but there is no such number in $\mathbb{R}$. This is possible because the nonexistence of this number cannot be expressed as a first order statement of the above type. A hyperreal number like w is called infinitely large; the reciprocals of the infinitely large numbers are the infinitesimals.

9. Construction of the Hyperreals

We are going to construct the hyperreals via sequences of reals. This is nice, because we can immediately identify the real number r with the an equivalence class of the sequence (r, r, r, ...), i.e. with [r, r, r, ...]$_\equiv$, and we can also add and multiply sequences:

$$[(a_0, a_1, a_2, \ldots)]_\equiv + [(b_0, b_1, b_2, \ldots)]_\equiv = [(a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)]_\equiv$$

and analogously for multiplication.

We may also compare sequences in terms of an ordering relation $\leq$, and there we run into trouble: some entries of the first sequence may be bigger than the corresponding entries of the second sequence, and some others may be smaller. We have to specify "which positions matter". Since there are infinitely many indices, we don't want finite sets of indices to matter. A consistent choice of "index sets that matter" is given by any free ultrafilter U on the natural numbers which does not contain any finite sets. Such an U exists by Zorn's Lemma(equivalent to the Axiom of Choice). (In fact, there are many such F, but it turns out that it doesn't matter which one we take.) We think of F as singling out those sets of indices that "matter": We write

$(a_0, a_1, a_2, \ldots) \leq (b_0, b_1, b_2, \ldots)$ iff the set of natural numbers $\{n \mid a_n \leq b_n\}$ is in F.

This is a total preorder and it turns into a total order if we agree not to distinguish between two sequences a and b if $a \leq b$ and $b \leq a$. With this "identification", the ordered field ${}^*\mathbb{R}$ of hyperreals is constructed. That is, we define an equivance relation:

$(a_0, a_1, a_2, \ldots) \equiv (b_0, b_1, b_2, \ldots)$ iff
$(a_0, a_1, a_2, \ldots) \leq (b_0, b_1, b_2, \ldots)$ or $(b_0, b_1, b_2, \ldots) \leq (a_0, a_1, a_2, \ldots)$

and an equivalence class:

$$[(a_0, a_1, a_2, \ldots)]_{\equiv} = \{(x_0, x_1, x_2, \ldots) \mid (x_0, x_1, x_2, \ldots) \equiv (a_0, a_1, a_2, \ldots)\}$$

The set of ***hyperreals*** $^*\mathbb{R}$ is then defined as the set of all such equivalence classes of real number sequences. The operations of an odered field may then be defined as indicated earlier. For example,

$$[(a_0, a_1, a_2, \ldots)]_{\equiv} + [(b_0, b_1, b_2, \ldots)]_{\equiv} = [(a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)]_{\equiv}$$

A hyperreat that is not a real is called ***non-standard***.

10. Infinitesimal and Infinite Numbers

A nonstandard real number *e* is called ***infinitesimal*** if it is smaller than every positive real number and bigger than every negative real number. Zero is an infinitesimal, but non-zero infinitesimals also exist: take for instance the class of the sequence (1, 1/2, 1/3, 1/4, 1/5, 1/6, ...). (This is a hyperreal because *F* contains all index sets whose complement is finite. Sowing that is it smaler than any non-zero positive real is more complicated.)

A non-standard real number *x* is called ***finite*** if there exists a natural number *n* such that $-n < x < +n$; otherwise, *x* is called ***infinite***. Infinite numbers exist; take for instance the class of the sequence (1, 2, 3, 4, 5, ...). A non-zero number *x* is infinite if and only if $1/x$ is infinitesimal.

Now it turns out that every *finite* nonstandard real number is "very close" to a unique real number, in the following sense: if *x* is a finite nonstandard real, then there exists one and only one real number st(*x*) such that $x - \mathrm{st}(x)$ is infinitesimal. This number st(*x*) is called the ***standard part*** of *x*. This operation has nice properties:

$\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y)$ if both *x* and *y* are finite
$\mathrm{st}(xy) = \mathrm{st}(x)\,\mathrm{st}(y)$ if both *x* and *y* are finite
$\mathrm{st}(1/x) = 1 / \mathrm{st}(x)$ if *x* is finite and not infinitesimal.
the map st is continuous with respect to the order topology on the finite hyperreals.
$\mathrm{st}(x) = x$ iff *x* is real

PART III. ORDER, COUNTING, CARDINAL AND ORDINAL NUMBERS[5]

1. Ordinal Numbers

In this section we approach the infinite anew to abstracts from familiar numbers to focus directly on the relation of infinity found in "numerical order" and "counting". In this section we discuss *order* and its primary main tool, *ordinal number*.

An infinte set is one that is ordered. In the order one element follows another. What makes the order infinite is that the order does not end. In this section we use set theory to construct a pardigm of such orderings. This order will exhibit these properties as little else. The details are familiar from the earlier construction of the natural numbers. We start the construction with a single stater element, called 0. In a general recursive manner we then specify what it is to add one more to the order to follow an element that is already there. All that is required is that the element be new, not the same as any previous element, and that it be placed next in the order. To do this we adopt the device of viewing the elements of the order as sets, identitfying 0 with $\varnothing$ and constructing mew elements in the order by means of the successor operation $\boldsymbol{S}$ defined as follows: $\boldsymbol{S}(x)$ is $x\cup\{x\}$. We then defined the ordering relation $\leq$ in terms of set inclusion:: $x\leq y$ if $x\subseteq y$.

Definition. The set ω of (***finite ordinals***) is defined as

1. $0\in\omega$,
2. for any x, if $x\in\omega$, then $\boldsymbol{S}(x)\in\omega$
3. nothing else is in ω.

Note that earlier ω was called the set of natural numbers $\mathbb{N}$, and later this same set will be identified with the and the first infinite cardinal number and called $\aleph_0$. As we know from the natural numbers, arithmetical operations are definable on the finite ordinals and these form a ring. We repeat the earlier results here reformulated to refer to ω so that we may compare them with the arithmetical properties of with those of later construction to follow.

Definitions Given $\boldsymbol{S}$, if c and g are defined, then f is defined *recursively* relative to $\boldsymbol{S}$, c and g, as follows: $f(0)=c$ and $f(\boldsymbol{S}(n))=g(f(n))$.

1. $\leq$ is a binary relation on ω (the ***less than*** relation) defined as follows: $x\leq y$ iff $x \subseteq y$. (By convention $x<y$ abbreviates $x\leq y$ and $x\neq y$.)

6. $+$ is a binary operation of ω (***addition***) defined (recursively):
   a. for all x in ω, $x + 0 = x$,
   b. for all x and $y\omega$, $x + \boldsymbol{S}(y) = \boldsymbol{S}(x + y)$
7. $\bullet$ is a binary operation (***multiplication***) of $\mathbb{N}$ defined (recursively):
   a. for all x in ω, $x \bullet 0 = 0$,

[5] See Raymond M. Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (Oxford: Clarendon, 1996).

b. for all x and y in ω, $x \bullet \boldsymbol{S}(y) = (x \bullet y) + x$

Definitions 2= $\boldsymbol{S}(1)$, 3= $\boldsymbol{S}(2)$, 4= $\boldsymbol{S}(3)$, etc.

Theorem

1. $\boldsymbol{S}$ is defined for all elements of ω,and $\boldsymbol{S}(x)\neq 0$,
2. for all x and y of ω, if $\boldsymbol{S}(x)=\boldsymbol{S}(y)$ then $x=y$,
3. **Mathematical Induction**) for any set A, if $0\in$A and, for any x, if $x\in$A then $\boldsymbol{S}(x)\in$A, then for any $x\in\omega$, $x\in$A)
   $\leq$ is a total ordering of ()
4. ω,
5. $<\omega,\leq,=,+,\bullet,0,1>$ where = is the identity relation on sets in ω is an ordered ring with identity relation =.

2. Infinite and Transfinite Ordinals

It has been observed since antiquity that there are cases in the natural world in which there seem to be both an infinite number of things and more that come after them. Consider the number of fractions distances between "here" and "there", as in Zeno's dichotomy paradox. Achilles goes half way, then half the distance of what of what's left, then half of that, etc. When you get gets there it seems that he must have traversed an infinite number of fractional distances, a possibility Zeno suggested was absurd. How are we to make sense of the idea that there are items in an order on the other side of an infinite number of prior elements? Infinite ordinal numbers are constructed to represent such orderings. We add a new element "beyond" all those in the set ω. To carry on with the construction use to define ω, the new element should be a set and it should come after all the elements in ω according to the ordering relation $\leq$, which is just set inclusion. We achieve this result if we simply identify the new element with ω itself. Because of the way ω is inductively defined in set theory by the successor operation, there are alternative ways to refer to it. Another name for ω is "the union of all elements in ω" (i.e. $\bigcup\omega$), and yet another way is "the union of all $\subseteq$-chains of elements in ω" (i.e. $\bigcup\{C \mid C$ is a $\subseteq$-chain of elements of $\omega\}$). Either of these two descriptions uniquely define the set of all its $\subseteq$-prior elements made up inductively by $\boldsymbol{S}$, and either may be used to define a general way to make up a new element to add to the earlier ordering. In the construction below we use the second. Once we have added ω to the ordering, we may then apply the successor relation to it and its successors: $\omega+1 = \omega \cup \{\omega\}$, $\omega+2 = \omega+1 \cup \{\omega+1\}$,$\omega+3 = \omega+2 \cup \{\omega+2\}$, etc. The result is one infinity following another:

$0,1,2,3,\ldots, \omega, \omega+1, \omega+2,\omega+3,\ldots$

Because we have a general method for generating a new element from such an ordering, we may add yet a further element beyond this double infinity, and then commence to take the successors of it. In this way we develop the hierarchy of what are called ***ordinal numbers***. It turns out that every set may be put into 1 to 1 correspondence with one and only one ordinal number, and the family of sets of that correspond in this way to an ordinal is called an ***order types***, and every set then falls into a unique order type. Ordinal numbers can therefore be used in a sense to "count" or "measure" other sets, and to rank sets by their "size" as that notion is defined by the construction in terms of the successor operation and the ordering relation $\subseteq$.

Definition. A family C of subsets is called a ***chain*** iff it is $\subseteq$-complete: i.e. for any y and z in C, $y\subseteq z$ or $z\subseteq y$.

Definition.

4. $0\in$ Or,
5. for any x, if $x\in$ Or, then $\boldsymbol{S}(x)$ (recall that $\boldsymbol{S}(x)=x\cup\{x\}$) ,
6. for any chain C of subsets of Or, $\bigcup C\in$ Or. (Here $\bigcup C$ is called a ***limit ordinal***).

Theorem. ω is the first (least) limit ordinal.

Theorem. Mathematical Induction on Ordinals.
If the following three conditions are met:

1. $0\in A$
2. for any x in Or, if $x\in A$ only if $\boldsymbol{S}(x)\in A$,
3. for any B if B is a union chains of elements of Or, $B\in A$

then it follows that Or$\subseteq A$.

Definition. For any ordinals x and y,

1. $x < y$ iff $x\subset y$,
2. $x \leq y$ iff $x\subseteq y$

Theorems

1. For any ordinals x and y in Or,
   $x < y$ iff $x\subset y$ iff $x\in y$.
2. For any ordinals x and y in Or,
   $x < y$ or $x = y$ or $y < x$.
3. For any x, if $x\in$ Or, then $x=\{y|\ y\in \text{Or}\ \&\ y<x\}$.
4. If λ is a limit ordinal, then for any ordinal x if $x<\lambda$, then $\boldsymbol{S}(x)<\lambda$.
5. **Counting Theorem**. For any A if $<A,\leq>$ is a well ordering, there exists a unique x in Or such that there is an f such that f is an issomorphism from $<A,\leq>$ to $<x,\leq>$

Definitions. Ordinal Arithmetic. Let x and y be in Or and let λ be a limit ordinal in Or.

1. Ordinal Addition +:
   - i. $x+0=x$
   - ii. $x+\boldsymbol{S}(y)= \boldsymbol{S}(x+y)$
   - iii. $x+\lambda=\bigcup_{y<\lambda}(x+y)$
2. Ordinal Multiplication • :
   - i. $x\bullet 0=0$
   - ii. $x\bullet\boldsymbol{S}(y)= (x\bullet y)+x$
   - iii. $x\bullet\lambda=\bigcup_{y<\lambda}(x\bullet y)$
3. Ordinal Exponentiation • :
   - i. $x^0=1$
   - ii. $x^{\boldsymbol{S}(y)} = (x^y)\bullet x$
   - iii. $x^\lambda=\bigcup_{y<\lambda}x^y$

Defintions. Let x be in Or.
$x+1=\boldsymbol{S}(x)$
$x+2= \boldsymbol{S}(\boldsymbol{S}(x))$

x+3= $\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{S}(x)))$, etc.

Theorem. Ordinal addition is not commutative, and ordinal arithmetic does not form a communative ring.
Proof. Let x be in Or. By definition $x+1=x\cup\{x\}$. Hence, $\omega+1=\omega\cup\{\omega\}$. But ω is a limit ordinal. Hence $1+\omega=\bigcup_{y<\omega}(x+y)=\omega$. But $\omega\neq\omega\cup\{\omega\}$. Hence $\omega+1\neq1+\omega$. **QED.**

Theorem. Let x, y, and z be ordinals in Or.
1. $0+x=x$
2. $x\leq x+y$
3. $x+(y+z)=(x+y)+z$
4. if $x\leq y$ then $x+z\leq y+z$
5. $\omega\bullet2\neq2\bullet\omega$
6. $1\bullet x=x\bullet1$
7. $x\bullet(y\bullet z)=(x\bullet y)\bullet z$
8. $x^1=x$
9. $1^x=x$
10. $x^{(y+z)}=(x^y)\bullet(x^z)$

3. Cardinal Numbers

Ordinal numbers exhibit many features of infinite sets. They are ordered, and even allow for one inifinity to be order aftr another. The also possess arithemetical operations that are normal (constitute a ring) on finite ordinals ad exhibit some of the familiar properties on higher ordinals. They also capture some concept of size because according to the Counting Theorem every set whatev er is issomorphic to some ordinal, and hence can be placed in "order" of "size" compared to any other set whatever. Moreover the ordinals up to ω consist of the counting numbers 0,1,2,… Georg Cantor adopted a straghtforard conception of counting: to count a set A is to place it in 1 to 1 correspondence to an ordered set of the counting numbers. Using the construction for the finite ordinals defined above n={0,1,2,,…,n-1}. To say then that a set A has n elements means that there is a 1 to 1 correspondence between it and the set n. This notion of counting can then be used to compare the "size" of sets. Two sets are the same size if they have the same number of elements. That is, if they cam be put in 1 to 1 correspondence to the same number and hence to each other. The idea is straightforwardly generalizable to sets of any size, including infinite sets. They are the same size, or equipollent iff they can be put into 1 to 1 correspondence.

Using this notion of "same size," however, Cantor discovered or, perhaps it is more accurate to say, reformulated in precise terms a "paradox" that has been known since ancient times: some infinite sets are bigger than others. Gallileo put the paradox this way. Draw a right triangle ABC with its vertex B at 0, A on the y-axsis and C on the x-axsis. Now for any point x on the segment BC there is unique point <x,y> on the hypotheuse AC, and conversely for any point <x,y> on AC there is the unique point x on BC. Hence there is a 1 to 1 correspondence between BC and AC. Hence AC and BC are the same length.

But the length of AC is hypotheuse of the triangle and hence its length is greater than that of BC.

This is not what a logician tems a genuine paradox because it can be explained. It is resolved by asknowledging the trueh of the proposition that some sets can be set in one to one correspondence with its proper subsets. For example is is a simple matter to define a 1 to 1 correspondence *f* between the complete set of positive finite ordinals greater than 0 and a proper subset, the even finite ordinals: $f(x)=2x$.

Indeed, the property of being equipollent to a proper subset seems to characteristic of infinite set. We know in a crude counting sense that the set {0,1,2,3,..} is infinite, and given the Cantorian notion of same size that every set equipollent to {0,1,2,3,…} is infinite. Moreover this set can be put into one to one correspondence with some of its proper subsets. This invites the generalization that an infinite set is characterize by the fact that it is equipollent to one of its proper subsets.

Cantor, moreover, discovered that not all infinite sets in this sense are equipollent. That is, using his notion of "same size" and "greater than" it follows that some infinite sets are larger than others. This result depends on two ideas, the notion of "same size" that we have already met and a new definition of "is smalle than." One finite set is smaller than another than another if they are equipollent finite ordinals that are ranked by the relation < on ordinals. But this criterion will not work for infinite sets. As the foloowing results shows a set that is bigger than another from the point of view of the ordinal construction in terms of successor and set inclusion can be the same size from the perspective of equipollence:

Theorem. For any finite ordinal (natural number) *n* in ω and any ordinal *x* in Or,

1. $\omega \approx \omega+n$ but
2. $\omega < \omega+n$

Proof. Part 2 was proved earlier. Part 1 is proven by off setting the pairing of ω with ω by *n* places.

What then is the appropriate ordering relation for size as measured by equipollence? Cantor discovered that not all infinite sets are equipollent. For example, he showed that infinite set is not equipollent to its powerset – the set of all its subsets. We shall review the proof below, but first let us see how this discovery leads to the appropriate sense of order among categories of equipollent sets. A set, we must grant, is at least as large as its own powerset because the set is a subset of its powerset – every element of the set is an element of the powerset as well. It follows then that an infinite set that is not equipollent to its powerset must be smaller than its powerset. Cantor makes this idea precise. One set is smaller that another if it is not equipollent to it but is equipollent to one of its proper subsets.

This notion of order generalizes from both from finite and infinite sets. Lets consider the case of finite sets first. Let *n* and *m* be finite ordinals such that $n<m$. Clearly if $n<m$, it follows on the earlier construction of ω that $n\subset m$ and that *n* cannot be put into 1 to 1 correspondence with *m.* Hence the smaller is not

equipollent to the larger but is equipollent to one of the larger's proper subsets, namely the smaller set itself. Consider more generally any finite sets *A* and *B* such that *A* is smaller than *B* because *A* is equipollent to some finite ordinal *n* and B to some finite ordinal *m* such that $n<m$. Then, there is a mapping *f* that is 1 to 1 from *m* onto *B.* Since Consider $n\subset m$, *n* is included in the range of *f.* Consider the set *C of* all values of *f* for arguments in *n.* (In the jargon of set theory *C* is *f* "*n,* the image of *n* relative to *f.*) Now *n* is also in 1 to 1 correspondence with *B* by, say, the mapping *g* from *B* to *n.* Clearly, *C* is a proper subset of *B.* Moreover, *C* can be put in 1 to 1 correspondence with *A* by a function *h* defined as follows: for any *x* in *B* define $h(x)=g(f(x))$. Consider now infinite sets. The notion that $A<B$ iff $A\subset B$ and *A* is not equipollent to *B* is well defined. It determines a strict pre-ordering (it is non-reflexive, asymmetric, and transitive), and we have examples to show the relation is non-empty. The more general definition that applies even to cases in which A is not a subset of *B* is therefore well defined and non-empty. It too determines a strict pre-ordering: $A<B$ iff for some *C, A* is equipollent to *C,* $C\subset B$, and *A* is not equipollent to *B.*

Definitions

1. *A* is ***the same cardinality as*** or is ***equipollent to*** *B*, abbreviated as $A\approx B$, iff there is a 1-1 correspondence between *A* and *B* (i.e. there is a 1 to 1 onto function with domain *A* to range *B*).
2. *A* ***has less cardinality than*** *B*, abbreviated $A\lessdot B$, iff for some C, $C\subset B$ and $A\approx C$, but it is not the case that ($B\approx C$)
3. *A* ***has the same or less cardinality than*** *B*, abbreviated $A\leqq B$ iff, $A\approx B$ or $A\lessdot B$

Theorem (Cantor). For any set *A*, $A\lessdot\mathfrak{P}(A)$

Proof. We show first that it is not the case that $A\approx\mathfrak{P}(A)$. We do so by a reduction to the absurd. To begin the proof, we assume the opposite, that $A\approx\mathfrak{P}(A)$. Then, there is a 1-1 mapping ***f*** from *A* onto $\mathfrak{P}(A)$. Now consider the set:

$B = \{x|\ x\in A\ \&\ {\sim}x\in \boldsymbol{f}(x)\}$.

Clearly *B* is a subset of *A*. Hence, since ***f*** maps *A* onto $\mathfrak{P}(A)$, there must be some *y* in *A*, such that $\boldsymbol{f}(y)=B$. Consider now two alternatives.

I. Suppose first that $y\in\boldsymbol{f}(y)$. Then, since $\boldsymbol{f}(y)=B$, we may substitute identities and obtain $y\in B$. But then by the definition of *B*, ${\sim}y\in\boldsymbol{f}(y)$. Hence, $y\in\boldsymbol{f}(y)\rightarrow{\sim}y\in\boldsymbol{f}(y)$.

II. Suppose the opposite, alternative, namely that ${\sim}y\in\boldsymbol{f}(y)$. Now, since $y\in A$ by hypothesis, *y* meets the conditions for membership in *B*, briefly $y\in B$. Then, since $\boldsymbol{f}(y)=B$, by Substitutivity of identity, $y\in\boldsymbol{f}(y)$. Hence, ${\sim}y\in\boldsymbol{f}(y)$ iff $y\in\boldsymbol{f}(y)$.

By I and II, it follows that $y\in\boldsymbol{f}(y)$ iff ${\sim}y\in\boldsymbol{f}(y)$. But this is a contradiction. Hence the original hypothesis is false, and we have established what we set out to prove, namely it is not the case that $A\approx\mathfrak{P}(A)$. There remain two possibilities: either $\mathfrak{P}(A)\lessdot A$ or $A\lessdot\mathfrak{P}(A)$. However, we can apply the argument above to any $B\subseteq A$, showing that it is not the same size as $\mathfrak{P}(A)$. Hence we may generalize that for all $B\subseteq A$, ${\sim}[B\approx\mathfrak{P}(A)]$. But logically, this fact entails that there no proper subset *B* of *A* such that $B\approx\mathfrak{P}(A)$. We have therefore eliminated the possibility that $\mathfrak{P}(A)<A$. It follows that the only remaining alternative must be true, namely that $A\lessdot\mathfrak{P}(A)$. **QED**

It is natural accodingly to investigate the properties of this new order $\leqq$ and the quivalence relation $\approx$. We do so by construct abstract entities that represent this order, the ***cardinal numbers***.

Definition. For any *x, x* is a ***cardinal number*** in Cr iff *x* is an ordinal in Or and there is not *y* such that *y* is an ordinal in Or such that $y<x$ and $x\approx y$.

Cantor labels the infinite cardinals $\aleph_0$, $\aleph_1$, $\aleph_2$, etc. $\aleph_0$ is pronounced "aleph null" or "aleph naught." (Note that $\aleph$ is the first letter of the Hebrew alphabet.) Sometimes $\aleph_0$ is abbreviated to just $\aleph$. Note that $\aleph_0$, aka $\aleph$, is just another name for ω the set of finite ordinals ω and for the set of natural numbers $\mathbb{N}$.

Definition. The set $\aleph$ is defined as

1. $0\in\aleph$,
2. for any *x,* if $x\in\aleph$, then $\boldsymbol{S}(x)\in\aleph$
3. nothing else is in $\aleph$.

Definitions.

1. *A* is ***infinite*** iff for some *B*, $B\subseteq A$ and $B\approx\aleph_0$.
2. *A* is ***countably infinite*** or ***denumerable*** iff for some $A\approx\aleph_0$.

Theorem. *A* is infinite iff for some *B*, $B\subset A$ and $A\approx B$.
(Note: the proof of this theorem depends on higher axioms of set theory.)

Lemma. For any *A* and *B,* there is some A' and B', such that $A\approx A'$, $B\approx B'$, and $A\cap B'=\varnothing$.

Theorem (Schöder-Berstein). For any *x* and *y* in Cr, if $x\leqslant y$ and $y\leqslant x$, then $x\approx y$.

Theorems. Let *A* and *B* be an arbitrary set.

1. If *A* is denumerable, then for some *B,* $B\subset A$ and $A\approx B$.
2. For any i=1,2,..,n, if B_i is denumerable, then $\bigcup\{B_i\}$ is denumerable.
3. If *A* and *B* are denumerable, then $A\times B$ is denumerable.
4. If *A* is denumerable, then A^n is denumerable.
5. If *A* is denumerable and *B* is finite, then $A\cup B$ is denumerable.
6. If *A* is infinite, then there exists a *B* such that $B\subseteq A$ and *B* is denumerable.
7. If *A* is infinite, then there exists a *B* such that $B\subset A$ and $B\approx A$.
8. *If *A* is infinite, then $A\times A\approx A$.
9. * If *A* is infinite, $A\leqslant B$, $B\neq\varnothing$, then $A\times B\approx A$.

Theorems.

1. (**Tricotomy**) For any *x* and *y* in Cr, either $A\leqslant B$ or $B\leqslant A$ or $A\approx B$.
2. For any *x* n Or, there is a *y* in Or such that $x\leqslant y$.

Definition. For any set *A*, $|A|$ = the one and only *x* in Cr such that $A\approx x$.

Theorems. Let *A* and *B* be an arbitrary set.

1. $A\approx B$ iff $|A|=|B|$
2. $A\leqslant B$ iff $|A|<|B|$
3. $A\leqq B$ iff $|A|\leq|B|$

Theorems. Let *x* and *y* be arbitrary cardinals in Cr.

1. $x \approx y$ iff $x=y$
2. $x \lessdot y$ iff $x<y$
3. $x \leqq y$ iff $x \leq y$

Definitions. Let *x* and *y* be arbitrary cardinals in Cr.

1. $x+y = |x' \cup y'|$ such that $x \approx x'$, $y \approx y'$, and $x' \cap y' = \varnothing$.
2. $x \bullet y = |x \times y|$
3. $x^y = |\{ f \mid$ such that f is a function from y into $x\}|$

Theorems (Transfinite Cardinal Arithemetic). Let *x, y* and *z* be arbitrary cardinals in Cr.

1. $x+y = y+x$
2. $x+(y+z) = (x+y)+z$
3. $x+0 = 0$
4. If $x \leqq y$ then $x+z \leqq y+z$
5. If $x \neq 0$ and $x \leqq y$, then $x \bullet y = y$
6. $x \bullet y = y \bullet x$
7. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$
8. $x \bullet 0 = 0$
9. $x \bullet 1 = x$
10. If $x \leqq y$ then $x \bullet z \leqq y \bullet z$
11. $\langle Cr, \leqq, \approx, +, \bullet, 0, 1 \rangle$ an ordered ring with identity relation $\approx$.

4. The Cardinality of Numbers Systems

It is possible now to apply the notion of cardinality to the traditional numbers systems developed in Part II. The naturals, integers, and rationals are all denumerable sets. The Reals have the same cardinality as the power set of ω. The issue of whether there is a set of intermediate cardinality between ω and its power set is an open question

Theorem. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are equipollent (have the same cardinality).

Theorem. The cardinality of the set of reals $\mathbb{R}$ is the same as that of the set of subsets of $\mathbb{N}$ (i.e. $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$) and is greater than that of the rationals, integers and natural numbers ($\mathbb{N} \lessdot \mathbb{R}$, $\mathbb{Z} \lessdot \mathbb{R}$, and $\mathbb{Q} \lessdot \mathbb{R}$).

Continuum Hypothesis. There is no *x* such that, $\mathbb{N} \lessdot x \lessdot \mathbb{R}$.

Generalized Continuum Hypothesis. There are no *x,y, and z* such that,

1. $\mathbb{N} \approx x$,
2. $x \lessdot y \lessdot z$, and $z \approx \mathbb{R}$.

Theorem (Gödel). The Generalized Continuum hypothesis is consistent with the earlier axioms of ZF.

Theorem (Cohen). The Generalized Continuum Hypothesis is independent of the earlier axioms of ZF inculding the Axiom of Choice (i.e. neither it nor its negation follow from the axioms of ZF).

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