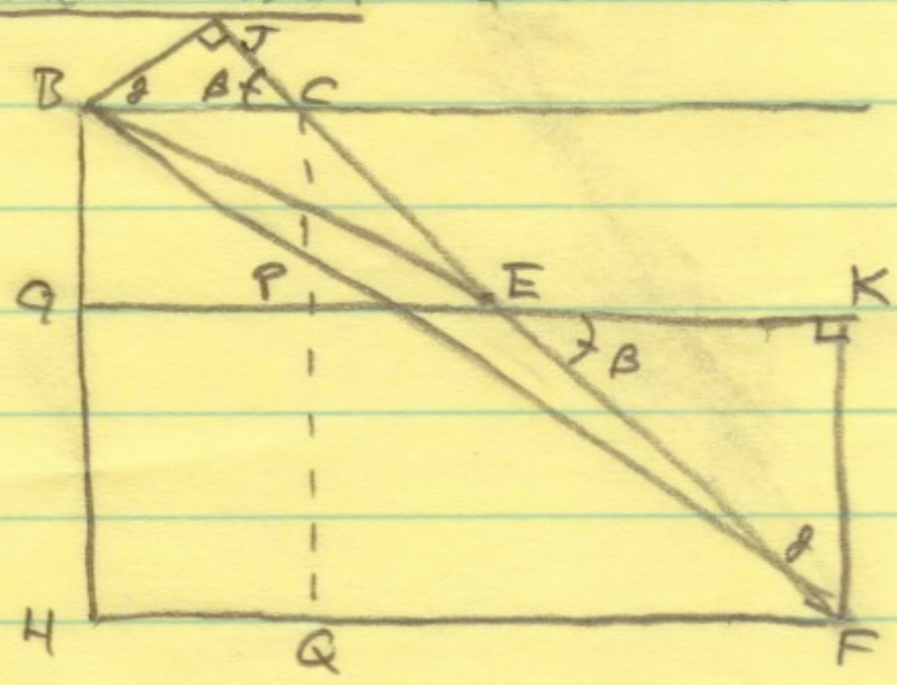
Notes on "Leibniz on the Quadrature of the Circle" (John Martin)

Bernard Aymon ["L'apport de Leibniz par la quadrature du circle"] Bulletin VSMP June 2002 *

Geometric Analysis



① $\Delta EKF \tilde{\sim} \Delta BJC$

$\therefore \frac{EF}{KF} = \frac{BC}{BJ}$, $EF \cdot BJ = KF \cdot BC$

② $PG = BC$ & $GH = KF$ by construction

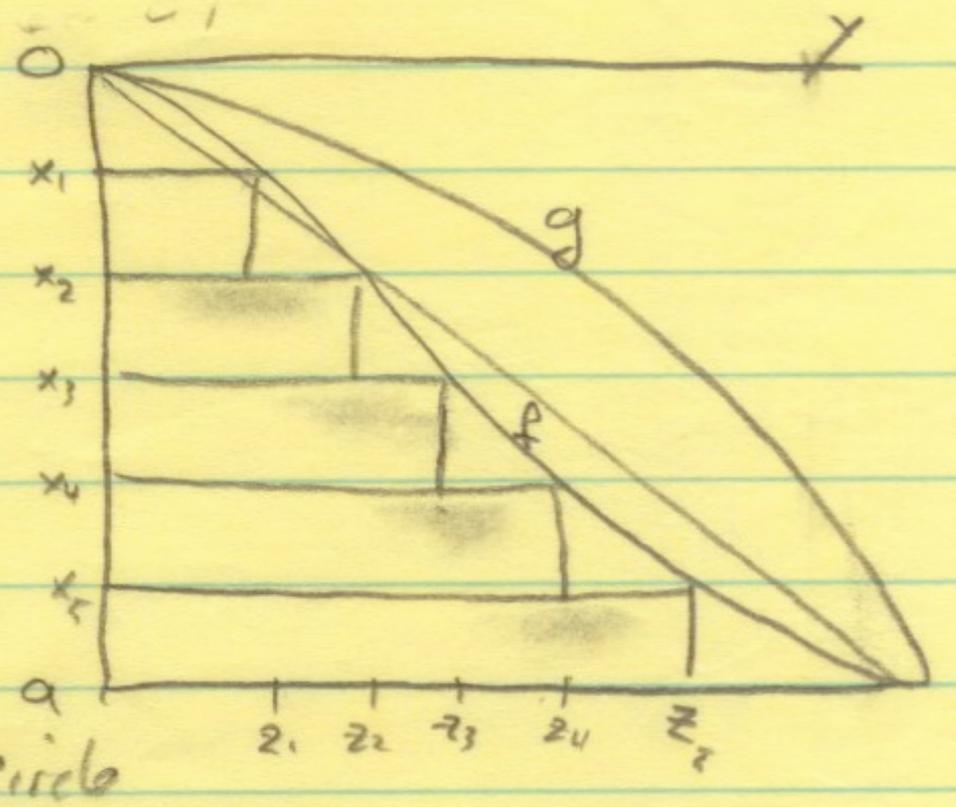
$\therefore KF \cdot BC = PG \cdot GH$

③ $EF \cdot BJ = PG \cdot GH$ by sub of =

$\therefore \frac{1}{2} EF \cdot BJ = \frac{1}{2}(PG \cdot GH)$

$\therefore$ Area $PGHQ = 2($Area $\Delta BEF)$

④ Area under g, i.e. $\int_0^x g(x)\,dx$



$= \frac{1}{2} ay + \frac{1}{2} \int_0^x f(x)\,dx$

$= \frac{1}{2} ay + \frac{1}{2} \int_0^x z\,dx$

Background Facts (Lemmas)

I. $\int_0^x z\,dx = xz - \int_0^z x\,dz$

II. $y = \sqrt{2ax - x^2}$ formula for a $\frac{1}{2}$ circle

$z_i = \frac{ax_i}{\sqrt{a^2 - (x_i - a)^2}}$ by pythagorean theorem

$\therefore x = \frac{2az^2}{a^2 + z^2}$ by theo 2 facts

III $\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 \ldots$

IV $\int z^n\,dz = \frac{1}{n+1} z^{n+1}$

* See also H. J. M Bos "Newton, Leibniz & the Leibnizian Tradition" I Grattan Guiness,

Theorem

Area under g (for the unit circle, $a=b=1$)

$= \frac{1}{2}ay + \frac{1}{2}\int_0^x z\,dx$

$= \frac{1}{2}ay + \frac{1}{2}(xz - \int_0^z x\,dz)$ by Lemma I

$= \frac{1}{2}(z(2a-x)) + \frac{1}{2}(xz - \int_0^z x\,dz)$ algebra

$= az - \int_0^z \frac{xa}{2a}\,dz$ algebra

$= az - \int_0^z \frac{az^2}{a^2+z^2}\,dz$ by Lemma II

$= b - \int_0^b \frac{z^2}{1^2+z^2}\,dz$ for $a=1$, $z=b$

$= b - \int_0^b z^2(1 - z^2 + z^4 - z^6 + \dots)\,dz$ by Lemma III

$= b - [\int_0^b z^2\,dz - \int_0^b z^4\,dz + \int_0^b z^6\,dz - \int_0^b z^8\,dz \dots]$ algebra

$= b - [\frac{1}{2+1}b^{2+1} - \frac{1}{4+1}b^{4+1} + \frac{1}{6+1}b^{6+1}]$ by Lemma IV

$= b - [\frac{b^3}{3} - \frac{b^5}{5} + \frac{b^7}{7} - \frac{b^9}{9}]$ algebra

$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$ for $b=1$

$= \frac{1}{4}$ area of unit circle (4 times Area under g)

$= \frac{1}{4}\pi 1^2$ formula for area of circle

$= \frac{\pi}{4}$ algebra

Corollary

$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} \dots$

Note continued: From Calculus to Set Theory, Princeton UP, 1980