Leibniz on Mathematics and the Actually Infinite Division of Matter

Samuel Levey

The Philosophical Review, Vol. 107, No. 1 (Jan., 1998), 49-96.

Stable URL:
http://links.jstor.org/sici?sici=0031-8108%28199801%29107%3A1%3C49%3ALOMATA%3E2.0.CO%3B2-3

The Philosophical Review is currently published by Cornell University.

The Philosophical Review, Vol. 107, No. 1 (January 1998)

Leibniz on Mathematics and the Actually Infinite Division of Matter

Samuel Levey

Mathematician and philosopher Hermann Weyl had our subject dead to rights:

> Among the heroes of philosophy it was Leibniz above all who possessed a keen eye for the essential in mathematics, and mathematics constitutes an organic and significant component of his philosophical system.[1]

Yet if it can hardly be denied that Leibniz is one of those philosophers who join metaphysics to mathematics, still, very little attention has actually been paid to the union of those two disciplines in his philosophy. Most recent trends in the study of Leibniz's metaphysics are finding it to be dominated by a theory of "broadly logical" concepts, or a philosophy of mind and action, or a theology, in which his mathematics plays at best only an incidental role.[2] But to miscast mathematics in that way is to misunderstand

---

I am very grateful to Christie Thomas, Jonathan Bennett, and Brent Mundy for comments on earlier versions of this paper, to Richard Arthur for discussion of some of the topics it treats, and to Walter Sinnott-Armstrong, Robert Sleigh Jr., and an anonymous referee for the *Philosophical Review* for their comments on a later draft.

[1]From the epigram to *Philosophy of Mathematics and Natural Science* (Weyl 1949, 2).

[2]This division of trends is a loose one, of course, and it does not divide commentaries into fully disjoint classes; but the trends are nonetheless worth observing. Russell 1900 and Couturat 1901 view Leibniz's metaphysics as based in his logic (more narrowly construed). The view of it as based in a theory of "broadly logical" concepts (to borrow Plantinga's phrase) such as *essence, necessity, definition, sufficient reason, individual notion,* etc., has found lots of favor throughout the past two decades, and there are too many valuable contributions to recite here, but at least the following ought to be mentioned: Mates 1986; Mondadori 1973, 1975; Sleigh 1983, 1990; and Adams 1983, 1985. Emphasis on philosophy of mind and action can be found in the works of (again) Sleigh 1990, Kulstad 1991, McCrae 1976, and Adams 1994. In his 1994, Adams also focuses on the impact of Leibniz's theology, as does Rutherford 1995. A further trend that might be mentioned would be the inspection of Leibniz's metaphysics from the point of view of his physics—see Arthur 1989, 1998; Garber 1982, 1985, 1995; Margaret Wilson 1976; and Catherine Wilson 1989. My own sympathies lie rather closely with this last camp, although the present paper is

Leibniz, for in fact his "philosophical system" is positively awash in the consequences of an interplay between mathematics and metaphysics that occurs at the very *center* of his thought and produces several of its most defining features. My project here is to begin to recover a fuller and, I hope, more properly balanced view of the design of Leibniz's metaphysics by bringing the role of that "organic and significant component" finally to light.

The general thesis of this paper is that Leibniz looks to mathematics in order to clarify his inquiries into the metaphysics of matter and in the process is forced to rewrite his mathematics (indeed, mathematics itself) at the foundations. But I intend also to paint a novel picture of the development of his thought that places emphasis on the very early writings—those of c. 1676, nearly thirty years before the *New Essays* and forty years before the *Monadology*—in which, I hold, his deepest views of mathematics and matter take shape together. Two of those writings I attend to rather closely: a dialogue on motion and the continuum that is perhaps one of the neglected masterpieces of early modern philosophy, *Pacidius Philalethei*; and a more enigmatic fragment on mathematics entitled "Infinite Numbers."[3] For in the *Pacidius* there emerges an analysis of matter and continuity that pointedly prefigures his celebrated later metaphysical views (c. 1705), and in "Infinite Numbers" Leibniz's unrecognized foundational studies of mathematics come to a head and yield his most considered philosophy of mathematics and philosophy of the infinite.

---

slanted towards Leibniz's mathematics much more than are any of those works emphasizing his physics.

[3]By contrast to their importance, both the *Pacidius* and "Infinite Numbers" are at present only barely visible in the scholarly literature. There are a few recent hints in their direction: Adams 1994 refers once to a passage in "Infinite Numbers," and Sleigh and Mercer 1995, a paper expressly offering a detailed survey of the early metaphysics, makes mention of the *Pacidius*. (Sleigh (1990, 114) remarks on the need for a close study of the problems of the labyrinth of the continuum in Leibniz's writings—problems especially featured in the *Pacidius* and "Infinite Numbers.") Three commentators should be singled out for having treated the *Pacidius* in published writings: Nicholas Rescher (1967); Catherine Wilson (1989); and most especially Richard Arthur (1989, 1998), whose fine forthcoming translations (*The Labyrinth of the Continuum: Leibniz's Writings on the Continuum, 1672–1686*, Yale Leibniz Series, vol. 2) should attract to the dialogue the careful attention it deserves. Also, Couturat includes the *Pacidius* in his 1903 collection of Leibniz's important but lesser-known writings (Leibniz 1903).

1. Backdrop: Cartesian Consequences of Motion in the Plenum

Descartes famously argues in his *Principles of Philosophy* that circular motion, and circular motion alone, could occur in a plenum without creating a vacuum. The special case of matter circulating through *irregular* spaces presents a complication: in order for the matter to fill now the greater, now the smaller spaces, it must travel proportionately faster through the smaller spaces and more slowly through the greater ones (AT 8A:59). Yet this is impossible, he says, "unless some part of that matter adjusts its shape to the innumerable different volumes of those spaces" (AT 8A:60).[4]

> And for this to happen, it is necessary that all imaginable parts of this piece of matter, which are in fact innumerable, should shift their relative positions some tiny extent. This minute shifting of position is actual division. (AT 8A,60)

It is worth pausing over whether a portion of matter has to suffer "actual division" in order to travel through all those spaces while at each moment still fitting perfectly into them. Given the example Descartes is considering, it seems that the "actual division" of a portion of matter into parts is supposed to imply a topologically *discontinuous* change in which cuts are imposed into the whole, or some sub-portions are shorn away from (what had been) some of their limiting points or boundaries. So the image of intraplenary

[4]I am responsible throughout for the translations of primary texts, but in translating Descartes I have consulted Cottingham, Stoothoff, and Murdoch 1985, and in translating Leibniz, Loemker 1969 and Ariew and Garber 1989. Also, for Leibniz's writings in A 6.3, I have consulted manuscripts generously supplied to me by Richard Arthur of his forthcoming translation volume (see n. 3 above). I abbreviate the primary texts thus: A = Berlin Academy Edition (Leibniz 1923–80); AT = Adam and Tannery, eds. (Descartes 1964–76); G = Gerhardt, ed., *Philosophischen Schriften* (Leibniz 1875–90); GM = Gerhardt, ed., *Mathematischen Schriften* (Leibniz 1849-63); *De Quadratura Arithmetica* = Knobloch, ed., *De Quadratura Arithmetica Circuli Ellipseos et Hyperbolae cujus Corolarium* . . . (Leibniz 1993); VE = *Vorausedition* to A 6.4 (Leibniz 1982–91). References to AT, G, and GM are to volume and page numbers; references to *De Quadratura Arithmetica* and VE are to page numbers; references to A are to series, volume, and page numbers. Also, I follow Arthur in adopting a notational convention (initiated by Edwin Curley) for translating the Latin term *seu*, which (like *sive*) corresponds to the English term 'or' where it implies equivalence: *seu* = or in other words, that is to say, etc. In the translations of the Latin, I mark the occurrences of this "'or' of equivalence" in Leibniz's text with a circumflex, thus: ôr.

motion at work here is this: while moving through those unequal spaces, matter fractures like glass into a vast multitude of tiny, shifting parts, though those parts stay packed together in such a way that still preserves the integrity of the plenum. But that view of what happens isn't strictly demanded by the stated constraints on the motion problem. What the Cartesian demonstration shows is only that a portion of matter circulating through unequal spaces in a plenum must "adjust its shape," that is, it has to suffer a *deformation*, and this fact does not support the inference to actual division or discontinuity. For while deformations can be topologically discontinuous, they can also be topologically *continuous*, and involve only twisting, stretching or squeezing without producing tears. If a portion of matter were thoroughly (maybe "metaphysically") *elastic* and could circulate through irregular spaces by deforming continuously, as, say, a greasy water balloon might be squeezed through a funnel into a wine bottle, then it wouldn't follow that actual division must occur.

It is possible, of course, that there are further unstated constraints concerning the metaphysics of matter or motion that could help support Descartes's inference to actual division. I'm inclined to think he tacitly takes matter to be, at its most basic level, radically *in*elastic, so that the idea of a truly continuous deformation simply has no foothold at all in the Cartesian physics. And he may, perhaps, be operating with a concept of "actual division" that makes it an essential feature of any such deformative change of shape. But to pursue the issue now would take us too far afield, and so I shall leave it merely raised to your attention.[5] Let us assume with Descartes that in adjusting its shape, matter actually divides into parts. As a consequence of matter's circulating through irregular spaces,

> there is an infinite or indefinite division of matter into small parts, and the number of these is so great that, however small a part of matter we may imagine, we must conceive it as undergoing actual division into still smaller parts. (AT 8A:60)

Having concluded that in order to circulate through such spaces in the plenum matter must actually divide into an infinity of parts,

[5]For useful discussion of Descartes on motion, see Arthur 1989 (to which the present account is indebted) and Garber 1992.

Descartes simply moves on to other topics, and in effect leaves behind what has the distinctive shape of a golden philosophical opportunity: solid grounds for a striking metaphysical thesis, and no follow-up analysis. In a sheaf of criticisms of the *Principles* published in 1692, one philosophical writer is to recall the argument of *Principles* 2:33–35 in essentially just that way:

> What Descartes says here is most beautiful and worthy of his genius. . . . Yet he does not seem to have weighed sufficiently the importance of this last conclusion. (G 4:370)

The writer is Leibniz, and the weight of "this last conclusion" is not lost on him. Leibniz completes his first close study of the *Principles* in December 1675, and in the watershed year of 1676 his philosophical inquiries into metaphysics and mathematics are embroiled in the hypothesis of the actually infinite division of matter.[6]

---

[6]Notice that in the preceding passage from Descartes, matter is said to suffer an "infinite *or indefinite* division." Descartes has two attitudes towards the infinite, one of which is very much finitist and constructivist and motivates his suggestion here that the infinite division of matter may be an indefinite (that is, indefinitely extensible) division rather than a division into an actual infinity of parts. Young Leibniz is rather impatient with this strand of Descartes's thought, and by 1671, in his *Theory of Abstract Motion,* he has already brushed off "indefinite division" (and presumably with it the latter's "indefinite number") as "being not in the thing, but in the thinker"(A 6.2:264). For Leibniz, then, the Cartesian demonstration establishes the existence of an actual infinity of parts and not simply an indefinite division of matter. Note also that, strictly, the occurrence of actually infinite division has been "established" only for *moving* matter; the portion of matter must be infinitely divided *during the interval of its motion* through unequal spaces. Might it sometimes be undivided and only potentially divisible, and actually divided only while moving in the relevant way? Since it is in fact a part of Leibniz's (ultimate) view—although an independent feature of it that I won't pursue in this paper—that matter is *always* suffering this sort of motion of its parts (for example, see A 6.3:565f., A 6.6: 58f.), the answer will somewhat trivially be no. A less trivial, negative answer also follows from the fact that Leibniz (like Descartes) strictly precludes from his ontology of matter the relevant concept of a "potential division." (While a body may be separable into various parts, that is not because that body contains "potential divisions" or "potential parts." Rather, a body is separable into various parts because it *actually* has contiguous parts that cohere together but which could be brought not to cohere and be separated from one another.) Potentiality, in the sense of potential divisions or potential parts, is a concept that belongs to the "ideal" realm of mathematics and geometry but has no application to the world of matter. See, for example, Leibniz's remarks at G 2:282, 3:622f., 4:492, 7:562f.; and see Hartz and Cover 1987 for a good overview.

2. A First Account of the Elements of Matter

To Leibniz's eye, the hypothesis of actually infinite division bulges with potential consequences for the metaphysics of matter. It is not hard to see why. The thought readily arises that if a finite portion of matter is to divide into an actual infinity of parts, those parts can be no larger than *points*. Indeed, this is precisely Leibniz's first reaction:

> It seems to follow from (the motion of) a solid in a liquid that a perfectly fluid matter is nothing but a multitude of infinitely many points, or bodies smaller than any that can be assigned. ("On the Secrets of the Sublime," February 1676. A 6.3:473)

These points, or at any rate, point-like "bodies," are characterized here as Leibniz often characterizes *infinitesimals*: the inassignably small—a magnitude greater than zero but less than any finite value you might specify.[7]

[7]As is well known by now, while an infinitesimal is on any account an inassignably small quantity in the sense just mentioned, the inassignable smallness of the infinitesimal can be taken in more than one way: roughly, as an intrinsic property of a fixed small magnitude (a "non-Archimedean" magnitude such that for any natural number $n > 0$, and infinitesimal i, there is no natural number k such that k times the infinitesimal i equals or exceeds n) or as a relational property of a variably assignable one. Though Leibniz is on occasion prepared to talk of the infinitesimal as "an infinitely small fraction or one whose denominator is an infinite number" (GM 4:93), this is not his most considered account of the concept. The most considered account, rather, is expressed when he writes of the infinitesimal as an "incomparable magnitude," a phrase he clarifies thus: "these incomparable magnitudes . . . are not at all fixed or determined but can be taken as small as we wish in our geometrical reasoning and so have the effect of the infinitely small in the rigorous sense" (Letter to Varignon, 1702. GM 4:91). He continues, in what is by then a characteristic refrain, "If an opponent tries to contradict this proposition, it follows from our calculus that the error will be less than any assignable error since it is in our power to make this incomparably small magnitude small enough for this purpose inasmuch as we can always take a magnitude as small as we wish." For a good discussion, see Ishiguro 1990, chap. 5; for an opposing account, see Earman 1975. It's hard to see how Leibniz, in imagining bodies as inassignably small, could be staying loyal to his proper view of the concept of an infinitesimal, for a body presumably is not the sort of thing that can itself "be taken as small as one wishes," but is a fixed, if still divisible, quantity. As I argue in "Leibniz's Constructivism and Infinitely Folded Matter" (forthcoming as Levey 1998), however, Leibniz is prone to allowing his constructivist thinking about mathematics to spread into his otherwise firmly actualist thinking about material reality, and sometimes

In imagining the point here as an infinitesimal body, Leibniz entertains one of three significant metaphysical readings of the concept of the point. The two most important others are the reading of the point as the metaphysical *minimum,* the indivisible unit "that has no parts" (A 6.3:540) and Euclid's definition of the point as an *extensionless simple,* or "that which has no part and no magnitude."[8] In a passage occurring just after the last one quoted, Leibniz explicitly contrasts the first two readings, the infinitesimal and the minimum:

> it must be rigorously examined whether there follows a perfect division of a liquid into metaphysical points, or only one into mathematical points. For mathematical points could be called Cavalierian indivisibles, even if they are not metaphysical points, ôr minima. (A 6.3: 474)

The "mathematical points" are the infinitesimals, and in using that name Leibniz is hinting at a possible link with the theoretical posits of one or another of the available infinitary mathematics, such as his own infinitesimal calculus, or Cavalieri's so-called "method of indivisibles." Yet already at the time of that writing, Leibniz is inclining towards taking the infinitesimals of his mathematics to be "fictions" with no claim at all to real existence, and this trace of the idea of actual infinitesimals in nature quickly vanishes.[9]

---

the resulting metaphysics is not coherent. Perhaps this is true in the present case, and Leibniz is allowing himself to treat inassignably small bodies not as vanishingly small, so to speak, but as *vanishing,* smaller and smaller—as small as you please. But for the purpose of our discussion, we can leave that aside, and think of the point-like bodies he mentions in A 6.3:473 above as non-Archimedean fixed quantities.

[8]Euclid's definition of the point occurs in *Elements,* bk. 1, df. 1. On Leibniz's terminology, see n. 11 below.

[9]The disappearance of the infinitesimal from Leibniz's serious philosophical thinking is a bit mysterious. After only the briefest flirtation with the idea of "real infinitesimals" in metaphysics (the "mathematical points"), he consistently avoids speculating about the composition of matter from infinitesimals, and repeatedly points out that the existence of infinitely small portions of matter does not follow from anything he says (GM 3:524, 535f.) I suspect that while Leibniz feels pressure not to commit himself to the existence of infinitesimal quantities, he is not fully satisfied that they have been proved to be impossible. The most popular grounds for denying the existence of infinitesimals, besides a lingering sense of their simply being conceptually "repugnant," are (1) the lack of any useful mathematical application, and (2) their incompatibility with the so-called axiom of Archimedes. (Archimedes' axiom states that, for any x and y such that $y > x > 0$, there is a natural number $n > 1$ such that n times $x > y$;

By "metaphysical points," on the other hand, Leibniz means partless, simple elements. (Notice that "mathematical points" or infinitesimals are *not*, as such, taken to be simple or indivisible, despite the term 'Cavalierian indivisibles'; that's why Leibniz contrasts them with "metaphysical points ôr minima.")[10] In his preferred terminology, metaphysical points are "minima," but the term is not strictly limited to points in Leibniz's use of it. Planes and lines, also, are sometimes counted as minima—presumably because planes are "minimal" in one dimension (they cannot be cut into thinner slices) and lines are minimal in two (they cannot be split into finer threads). Metaphysical points alone are perfect minima: they are indivisible in every respect.

The contrast between Leibniz's minima and Euclidean points holds some interest. Euclid's definition of a point differs from that

---

and, as its name suggests, it is widely accepted as axiomatic in early modern mathematics.) By Leibniz's lights, of course, "infinitesimals can at least be used in the calculus and in reasoning" (GM 3:535), so (1) lacks force for him. As regards (2), the axiom of Archimedes is so directly a denial of the existence of infinitesimals, it is hard to see that it supports any substantive argument for rejecting them. If infinitesimals are to be rejected, and the axiom upheld, some other, logically prior considerations ought to be brought to bear. At any rate, it is perfectly clear that by the spring of 1676 Leibniz takes the infinitesimals of his mathematics to be "fictions." And in writings of many years later, he occasionally suggests that he has a positive argument against the very possibility of infinitesimal quantities (GM 3:524, 551). I have so far been unable to find such an argument. I suspect that the argument, if there is one, must be somewhere in his (extensive) mathematical writings of 1676, probably occurring in mid-March.

It is worth noting that the concept of the infinitesimal is quite independent from that of the minimum. Being infinitely small carries no conceptual bar against being divisible into parts. And it is clear that Leibniz does *not* take his reasons for rejecting *minima* as counting against infinitesimals. In a letter of late August 1698, he writes to John Bernoulli, "just as there is no numerical element, ôr minimum part of the number one, or minimum number, similarly there is no minimum line, ôr linear element, for a line, like the number one, can be cut into parts or fractions. I admit that the impossibility of our infinitesimals does not follow directly from this . . . since a minimum is not the same thing as an infinitesimal" (GM 3:535–36).

[10]The title 'indivisibles' is an unfortunate one for the infinitely small entities postulated by Cavalieri's mathematics, and is positively misleading for Leibniz's infinitesimals. In neither case is indivisibility a property that the mathematical practice actually trades on, and in Leibniz's case, at least, infinitesimals are expressly said *not* to be indivisible (recall his "second-order" differentials, infinitesimals infinitely smaller than those of the first order). And as 'indivisible' is far more apt for the concept of the minimum, its application to infinitesimals tends to generate confusion.

of Leibniz's minimum only by its extra demand that the point have no *magnitude* as well as no parts.[11] The Euclidean point is by definition absolutely extensionless or of measure zero, whereas the metrical properties of minima are not specified at all. Thus while a Euclidean point may consistently be supposed to satisfy the concept of the minimum, so also may an *extended* simple, say, a Democritean atom. It would take a further, substantive thesis to link the concept of the partless unit to the concept of the extensionless unit, and although Leibniz would not necessarily be unfriendly to such a thesis, he does not wield any claim to that effect in places where it could fairly obviously be put to use. Indeed, in the lines of argument where the point (minimum) has its most important role in the study of the nature of matter, Leibniz remains, if not exactly neutral, at least indifferent to its metrical properties. Specifically, in *Pacidius Philalethei,* the arguments concerning the composition of the continuum from points are apparently tailored not to trade on a particular answer to the question whether points must be extended or instead extensionless.[12] For the purposes of advancing his inquiries into the metaphysics of matter, the point is simply the minimum (and hereafter I shall use 'point' and 'mininum' interchangeably).

---

[11]My reading of the minimum is controversial. It is widely believed among Leibniz's commentators that, in his hands, the concept of the minimum is the same as that of an extensionless Euclidean point, namely, "that which has no part and no magnitude." I disagree. In *Theory of Abstract Motion* (1671), Leibniz indeed uses the term 'minimum' to express the Euclidean concept of something "of which there is no magnitude or part," and goes on to deny vigorously the existence of such an entity (G 4:228f.). But as early as April of 1676 (and consistently thereafter), Leibniz is found calling the point simply that "of which there is no part" [*cuius pars nulla est*] (A 6.3:498); and the connection with the use of 'minimum' in his mathematics would suggest that a minimum is a *least* magnitude, but not a non-magnitude. Also, his most considered lines of argument involving minima occasionally positively require supposing that minima have some extension (though no particular extension). I am inclined to think that if Leibniz adds anything to the concept of simplicity in his idea of a metaphysical minimum, it is the concept of an element whose size, *whatever it may be (zero or otherwise),* fixes the absolute lower bound on the range of sizes that it is metaphysically possible for anything to have. (In that case, all "metaphysical points" would have the same magnitude, all "metaphysical lines," so to speak, the same breadth, and all "metaphysical planes" the same depth.) Nothing essential to the argument of the present paper, however, turns on these details of my interpretation of the minimum.

[12]See section 3 below.

To portray matter as resolving into points is the way Leibniz makes sense of the idea that matter is divided into an actual infinity of parts. It is his metaphysical spin on the idea of infinite division. The significance of his proposal must not be missed: if matter divides into an actual infinity of parts, *the ultimate parts of matter are points.* They are its "first elements," the ontological foundations out of which matter arises. Note that this holds equally for solid matter as for any "perfect fluid," for the "cement" that makes for solid matter (namely, motion)[13] is impermanent, and "insofar as matter exists when it ceases to have a cement . . . it will be reduced to a state of liquidity or divisibility from which it follows that it is composed of points" (A 6.3:473). All of matter, liquid or solid, is apparently "nothing but a multitude of points."

In the ensuing months, however, Leibniz abandons the picture of matter as resolved into points, for two reasons. First, in the course of crafting a new metaphysical analysis of the continuum, he comes to a view of points according to which they could not be *parts* of anything. Second, he develops a fresh mathematical understanding of how an infinite partition of a finite magnitude might be structured in such a way that matter could be divided into an actual infinity of parts without any of those parts being points. These topics provide the two foci of this paper, and we shall turn first to the lesser one: Leibniz's new analysis of the continuum and its attendant metaphysics.

3. Minima and the Labyrinth of the Composition of the Continuum

Starting about halfway through *Pacidius Philalethei,* Leibniz launches an argument for the thesis that the continuum could not be composed of points. If it were to be composed of points, he suggests, it would have to be composed either of a finite number of points or else of an infinite number of them,[14] and each alternative can be shown to be impossible.

[13]Motion, he says, is responsible both for unifying matter's parts and for dividing them. This is an essentially Cartesian idea (see *Principles* 2:25/ AT 8:53–54) that Leibniz explicitly adopts (see *Propositiones quaedam physicae,* prop. 14, A 6.3:28).

[14]Leibniz has his interlocutors open the argument thus: "*Pacidius*: What we must ask first of all is, do you compose a line or finite magnitude from a finite or from an infinite number of points [*ex finito an infinito punctorum numero*]? *Charinus*: Let us try composing it from a finite number" (A 6.3:

Take first the case of a finite number of points. It is a theorem of geometry that "any line whatsoever can be divided into a given number of equal parts," and so, in particular, any line can be divided into one hundred equal parts.

> [F]rom this I can already see that it is impossible for a line to be composed of a finite number of points. For on this supposition there will be some line, at any rate, that can be conceived to consist of ninety-nine points, a hundredth part of which certainly cannot be conceived without some fraction or aliquot part of a point. (A 6.3:549)

And to postulate a "fraction or aliquot part" of a point, that is, a proper part of a point, is flatly contrary to the hypothesis that points are *minima*. The continuum, therefore, could not be composed of a finite number of points. This arm of Leibniz's argument is cogent. What it shows, in essence, is that the hypothesis that a continuous object might be composed of a finite number of points cannot cope with the fact that the continuum is *infinitely divisible*. As Leibniz cashes out infinite divisibility in this context, the line has the capacity to be divided to "a given number"—that is, *any natural number*—of equal parts. This assures that whatever number is assigned as the number of points in the line, the line can be divided to a *greater* but still finite number of equal parts. If the number of parts cut out of the line is greater than the number of the points that constitute it, it follows that there are some parts of the line that are not *whole* points. As there is nothing besides points in the line to constitute its parts, however, those parts of the line that are not whole points must be constituted by "fractions or aliquot parts" of points. Again, this consequence forces the original hypothesis into a contradiction by requiring the existence of parts of "what has no parts," namely, the minimal point.

Taking up now the second arm of his argument, we suppose the continuum to be composed of an *infinite* number of points. From this supposition, Leibniz says, one can show that the continuum has a proper part that is composed of the same number of points and is therefore "equal to" the whole. That consequence, however,

---

548). It is in fact part of Leibniz's considered view, and one the interlocutors shortly reach in the dialogue, that *numeri infiniti* are impossible. Various forms of the expression 'infinite number' continue to occur in his writings even after his view is firmly settled, but always in contexts in which no serious commitment to infinite numbers is at stake.

is impossible, for it violates the "axiom" that the part is less than the whole. For his demonstration, Leibniz stays with the example of the continuous line. He draws out a first line and then intersects it with a diagonal. From points on the diagonal he drops lines to the segment of the first line lying below it (A 6.3:549f). Lines can be dropped in this way from each point on the diagonal to some point on the lower line-segment, in such a way that all their points are put in one-one correspondence. What this shows, of course, is that there are just as many points in the line-segment as in the diagonal. But the line-segment is shorter than the diagonal, and in fact is isomorphic with one of the diagonal's proper parts. So, the diagonal has some proper part comprising just as many points as the diagonal itself comprises. The part and the whole of the diagonal would thus have to be, in this respect, equal. But this violates the part-whole axiom; so the line, and more generally, the continuum, could not be composed of an infinite number of points.

Leibniz's line of argument here actually descends from Galileo's discovery of the fact that the elements of an infinite collection can be put in one-one correspondence with those of some of its own proper sub-collections (A 6.3:550f.). Each natural number, for example, has a unique square, and so there is a one-one mapping of the naturals onto the square numbers, each natural n going to a unique n^2 (1 goes to 1, 2 to 4, 3 to 9, etc.). The existence of such a mapping satisfies a sufficient condition for the class of naturals to have the same number of elements as the class of squares, and so, in this sense, the class of naturals is "equal to" the class of squares. (That phenomenon is of course not restricted only to the case of numbers. It is a remarkable feature of the infinite collection in general that its elements can be mapped one-one onto those of some of its own sub-collections. Indeed, Dedekind declares this feature to be the very hallmark of the infinite collection.)[15] Yet the class of squares is manifestly only a part of the class of naturals, and a vanishingly small part at that.[16] Thus, it seems that there is

[15] Dedekind's most famous discussion of this point is *Was sind und was sollen die Zahlen?* 1887, translated as "The Nature and Meaning of Numbers," in Dedekind 1901.

[16] As Galileo and Leibniz are well aware, the relative portion of squares to naturals falls off very dramatically as the naturals increase: while half of the first four naturals are squares, only a tenth of the first hundred are

a part of the class of naturals that is equal to the whole, a result that would contradict the axiom that the part is less than the whole.

This apparent failure of the part-whole axiom in the domain of the infinite is *Galileo's paradox,*[17] and it is properly a paradox of the infinite rather than of the continuum. Leibniz aims to pin the hypothesis that the continuum is composed of infinitely many points as incoherent by showing that it engenders Galileo's paradox. In a continuum—say, a line—composed of an infinite number of points, there would be a part whose component points can be mapped one-one onto those of the whole. Hence, the part and the whole of the line would have exactly the same number of component points. And so, in exactly that sense, there would exist a part of the line that fails to be less than the whole, which is impossible. Q.E.D.

At just that point, however, the force of Galileo's paradox sharply tapers off. For it is crucially unclear that the criterion according to which the part *fails* to be less than the whole is in fact the same criterion for "less than" that makes the part-whole axiom true.[18] No doubt the part must in *some* sense be less than the whole that contains it, but it is far from obvious whether that sense requires the criterion that the part be composed of a lesser number of points than the whole. By the criterion of *congruence,* say, the part of the line is less than the whole, for it is smaller than the whole. But by the criterion of *one-to-one correspondence of component points,*

---

squares, and only a *thousandth* of the first million, etc. See especially Leibniz's notes on Galileo's *Two New Sciences* (A 6.3:168).

[17]So called for Galileo's discovery and discussion of it. See his *Discourses and Demonstrations Concerning Two New Sciences,* in Galilei 1898 (70–80).

[18]Galileo's own reaction to the paradox is simply to reject the part-whole axiom in the case of the infinite whole; more exactly, he denies that the terms 'greater', 'lesser', and 'equal' apply in such cases. In criticizing the related case of Leibniz's argument against the possibility of an infinite *number,* Russell disparages the axiom, calling it "one of those phrases that depend for their plausibility upon an unperceived vagueness" (1919, 81); but as he essentially goes on to distinguish a particular sense in which the part may be said to be "equal to" the whole, I consider his strategy in defusing Galileo's paradox to be much the same as ours. (Note also that in his 1900, Russell *praises* what are tantamount to appeals by Leibniz to Galileo's paradox as "very solid arguments" against the hypothesis of an infinite number (109), and says that Leibniz's principle "that infinite aggregates have no number" is "perhaps one of the best ways of escaping the antinomy of infinite number" (117). See section 6 below for discussion of this facet of Leibniz's philosophy of mathematics.)

the part of the line is not less than the whole, but is rather equal to it. It is thus entirely possible that the part-whole axiom simply invokes a criterion for "less than" distinct from the one that the part and whole of the line fail to satisfy. If so, then it has not yet been shown that the hypothesis that the line is composed of an infinite number of points actually violates the part-whole axiom. It may be that the axiom is taken by Leibniz to hold necessarily for any reasonable construal of the "less than" relation, but I am not aware that he offers any claims to that effect—perhaps it never occurs to Leibniz to weigh carefully the idea that "less than" might be polysemous, invoking different criteria in different contexts. In any event, I doubt whether anything very satisfactory can be said on behalf of the claim that the part-whole axiom must hold for all reasonable construals of "less than."

Thus the two arms of Leibniz's argument against the composition of the continuum from points achieve quite different degrees of success. While he shows convincingly that no finite number of points could compose the infinitely divisible continuum, his case against composing the continuum from an infinite number of points is not cogent.

Leibniz, of course, is more enthusiastic about the success of both arms of his argument, and he concludes that, as neither a finite number nor an infinite number of points could compose it, the continuum "can neither be resolved into nor composed out of points" (A 6.3:555). He then glosses the finding that minima could not compose the continuum with a new analysis of both. Minima, he says, are not parts of things at all, but are rather *modes* of them, their "extrema" or boundaries.[19] They are the vertices, edges, and

[19]Note that the extrema of extended things are, on Leibniz's account, always intrinsic to them, in the sense that any extended thing is identical to its *topological closure.* Leibniz also uses 'extrema' to designate the *terminal elements* in numerical series. As we shall see below in section 7, he holds that infinite numerical series will lack at least one extremum—as, for instance, there is no last term in the series $\frac{2}{3}$, $\frac{2}{9}$, $\frac{2}{27}$, $\frac{2}{81}$, etc. And although we can still identify 0 as the *limit* of that series (and in that sense an "extremum" of it), it does not occur as an element within the series. Hence, unlike extended *things,* infinite numerical series will not be identical to their topological closures, for they will fail to contain at least some of their "extrema." (A referee for this journal made the nice suggestion that we might distinguish between "intrinsic" and "extrinsic" extrema or boundaries; thus, we might say that while numerical series may have one or more extrinsic extrema, extended things, on Leibniz's view, must always have intrinsic extrema.)

surfaces of *extensa* rather than their parts or limiting instances. As modes of objects rather than objects in their own right, minima depend for their existence upon the condition of the continuum. They result from the continuum's being divided a particular way into parts.

> [T]here are no points before they are designated. If a sphere touches a plane, the locus of contact is a point; if a body is intersected by another body, or a surface by another surface, then the locus of intersection is a surface or a line, respectively. But there are no points, lines or surfaces anywhere else, and there are no extrema in the universe except those that are made by a division. (A 6.3:553)

And the new analysis of the continuum does not stop at that. Not only do the minima fail to be prior to the continuum:

> Nor are there any *parts* in a continuum before they are produced by a division. (Next line in *Pacidius*; italics added.)

A continuum simply has *no* "first elements" or "foundations" among its parts, whether they be mythical minimal parts or simply lesser continua, for the continuum is not a true composite of its parts in the sense of being the ontological upshot of their aggregation. To the contrary, "in the continuum, the whole is prior to its parts" (A 6.3:502). To hold the continuum to be a composite of more basic elements—much less to be a composite of *minimal* elements—gets the order of priority exactly reversed, and worse it confuses the continuous with the discrete, for in discrete things "the whole is not prior to the parts, but the converse" (A 6.3:520).

That account of the continuum should have a familiar ring to readers of the later Leibniz. In particular, the analysis of continuous quantity as foundationless and prior to its parts, the emphasis on the distinction between continuous quantities and discrete quantities (whose parts are prior to the whole), and the claim that points are never parts but only *extrema* or termini of extended things, all pointedly prefigure critical strands of the analysis of the so-called labyrinth of the continuum that he offers in his later and most considered metaphysics (c. 1705).[20] That later metaphysics is,

---

[20]I have in a mind here an interval of seven years or so that includes and surrounds 1705: before the Des Bosses correspondence, and after his "New System of Nature" (roughly, the period of *De Ipsa Natura*, his correspondences with Bernoulli and de Volder, and his writing the *New Essays*.) For brevity, I shall call the metaphysical views of this vaguely bounded period "the 1705 metaphysics."

of course, his distinctive brand of panpsychism, featuring an ontological groundfloor of *monads*: indivisible, immaterial units, whose properties are various forms of striving and perceptual states, and which are the basic elements out of which all else, including material reality, is constructed. But to trace the connections between the 1676 and 1705 metaphysics, we must leave aside the monads for now and concentrate instead on the cluster of other concepts he uses to escape the labyrinth of the continuum.

In his later analysis of the continuum, Leibniz contends that the labyrinth arises from confusing the character of matter with that of mathematical or geometrical continua: either supposing matter to be continuous or supposing continua to be composites of discrete parts (G 2:282). (Contrary to a fairly common misconception, Leibniz does *not* treat the problems as being generated by "the assumption that unextended points can somehow 'sum up' to produce an extended being."[21] As I mentioned earlier, his arguments concerning the composition of the continuum are typically silent about the metrical properties of points.) The signature statements of his analysis of the continuum problems persistently refer to this confusion of the nature of the continuum with the nature of matter as "the confusion of *the ideal* with *the actual*" (G 4:419; italics added). It is here at the boundary between "the actual" and "the ideal," that is, the world of matter and the realm of mathematical and geometrical objects (and other abstract structures), that the significance of the relative priority of the part and the whole comes into play.

What sets the actual objects of the world of matter apart from

[21]The quote is from Rutherford (1990, 52). But note also that Rutherford cites this as just one of two "conceptual errors" that Leibniz thinks is responsible for generating the labyrinth; for he is quite correct in citing the assumption that "matter has the character of the mathematical continuum" as a culprit. Cf. also Catherine Wilson: "A continuous line, as Leibniz recognized, could not be generated from . . . extensionless points" (1989, 75). It is not difficult to project this view of the problem when reading Leibniz's text, for the mystery of how an *extensum* could consist of extensionless points is both intuitively gripping and obviously relevant to the composition of the continuum; and it has enjoyed some very interesting contemporary discussion (see, e.g., Grünbaum 1973 and Skyrms 1983). But it is a projection, nonetheless, to attribute it to Leibniz, for although he clearly thinks the proposition that an *extensum* could be composed of points is worthy of protracted refutation, so far as I am aware he never poses the difficulty in that particular way.

the ideal objects of mathematics and geometry is the difference between the *discrete* and the *continuous,* a difference Leibniz construes in metaphysical terms and one he believes is heavy with metaphysical consequences. Actual wholes such as material bodies are discrete, for they are aggregates that have a unique decomposition into "actual parts" that are "distinguished." Ideals, on the other hand, are continuous objects whose "essence is indeterminacy" and whose parts are "indefinite" or "indeterminate"(G 2: 282, 4:394, 7:563, 2:268). Or, to capture the contrast with "actual parts": the parts of an ideal whole are merely *possible* or *potential* parts. And crucially, it is "*because* the part is only possible and ideal" that the whole is prior to the part in the continuum (G 4:491f.).

Continuous wholes "are not really aggregated from parts, since there are no limitations at all on how one might wish to assign parts in them," for continua are, in themselves, indifferently divisible, and "signify nothing but the mere possibility of parts [taken] any way one likes" (G 2:276). Whereas an actual whole resolves in a perfectly determinate way into parts "in accordance with how nature has actually instituted divisions and subdivisions" (G 2:268), an ideal whole *abstracts* from the actual assignment of parts and only shows the possible ways that parts might be assigned in it. Each way of assigning parts expresses one possible decomposition of the whole, or one way that it is possible for parts of matter to hang together and form a composite object. With those distinctions in place, Leibniz's analysis of the problem becomes clearer in another occurrence of its signature statement:

> As long as we seek actual parts in the order of possibles and indeterminate parts in the aggregates of actuals, we confuse ideals with real substances and we entangle ourselves in the labyrinth of the continuum and inexplicable contradictions. (Letter to de Volder, 1706. G 2: 282)

Only two aspects of the later analysis are as yet unaccounted for in Leibniz's early writings: the claim that it is a *confusion* that generates the labyrinth and its paradoxes, and the claim that continua are only ideal, *entia rationis,* whereas discrete things are actual or real. Both claims, however, are visible in a striking 1676 passage:

> modes of substance themselves, if they are confused with similar entities of reason or aids for the imagination, can never be rightly understood. . . . To see this still more clearly, take this example. Whenever someone has conceived duration abstractly, and by confusing it

> with time has begun to divide it into parts, he will never be able to understand how an hour, for example, can pass. For in order for the hour to pass, it will be necessary for half of it to pass first, and then half of the remainder, and then half of what remains of this remainder. And if he subtracts half from the remainder in this way indefinitely, he will never be able to reach the end of the hour. Hence many people who have not been in the habit of distinguishing *entia rationis* from real things, have dared to declare that duration is composed of moments. In their desire to avoid Charybdis, they have run into Scylla. (A 6.3:279)

Confusion between "entities of reason" and "real things" leads one to posit discrete parts (such as moments) in what are in fact continuous wholes (such as duration "conceived abstractly"), and quickly embroils one in the classical paradoxes of the continuum. One especially remarkable feature of this early anticipation of Leibniz's later analysis of the source of the labyrinth is that the author of the passage just quoted is *Spinoza.* Leibniz here is simply recopying an excerpt of the so-called Letter 12, *On the nature of the Infinite,* of 1663 (Geb 4:57/17–58/15), but the uncanny resemblance between the excerpt's apparent import and aspects of Leibniz's own later views strongly suggests that those aspects originate in Leibniz's 1676 Spinoza studies.[22]

Even if his Spinozistic claim that the labyrinth arises from a confusion between the real and the ideal is correct, and those embroiled in the labyrinth are confusedly searching for discrete, determinate parts in continuous, indeterminate wholes, it still needs to be asked, what is the *basis* of this difference between the actual and the ideal, the discrete and the continuous? Leibniz proposes

[22]So far as I know, no one has taken up this connection between Leibniz's and Spinoza's thoughts on the paradoxes of the continuum. It remains, however, a delicate question whether Spinoza fully anticipates, or indeed educates, Leibniz on this score, for it is a matter of scholarly debate what exactly Spinoza has in mind by 'duration' and 'time', and therefore what he intends by the remarks we quoted from Letter 12. Donagan's reading (1988, 109–11) of Spinoza's concept of time—both *duratio* and *tempus*—would not cohere with a Leibnizian gloss, so to speak, of the one as undivided and continuous, and the other as composite and discrete. By contrast, Bennett's reading (1984, 202–5) would make Spinoza out to be drawing a Leibnizian distinction here between the discrete and the continuous, but to be reversing the account of the real and the ideal: the discrete is a mere entity of reason, whereas the continuous is real. (It is also an open question what exactly Leibniz takes Spinoza to be saying.) I shall not enter into this controversy in Spinoza scholarship in the present paper.

that the difference points up a great disparity between their respective metaphysical foundations. The determinacy of discrete objects owes to the presence of "real substances" that, as he says, "are in" actuals, by which he means that actuals are constructed out of, or founded on, "the multitude of monads ôr simple substances"(G 2:282).[23] The world of matter decomposes in a particular way into parts, because there is a real underlay of substances that it is constructed out of in a particular way. The indeterminacy characteristic of continuous objects, on the other hand, is due precisely to the *absence* of any monadic substructure. There are "no limitations at all on how one might wish to assign parts" in continua because there is *nothing back there* constraining how those parts are to be assigned. Continua, unlike material objects, are pure constructs of the mind—ideals, *entia rationis*—and in no way actually incorporate substantial reality.

A remark written in February 1676 offers a nice backdrop against which to consider the views of 1705 that we have just been sketching, and their prefigurement in the writings of later months of 1676. In February, Leibniz is still prepared to entertain the idea that

> one could defend the composition of liquid [matter] out of perfect points, even if it is never absolutely resolved into them, on the grounds that it is capable of all resolutions. ("On the Secrets of the Sublime." A 6.3:474)

On this picture, matter retains features perhaps naturally conceived as belonging to what in the study of geometry is "the continuous manifold," and out of which geometrical objects are constructed. Matter, like the manifold, is composed of points and "capable of all resolutions," and as a multitude of points it remains *essentially* indifferent to the actual assignment of those points as parts of particular bodies (at least in the sense that those very points would still exist even if all of matter were "resolved" in some different way into a totally different sequence of particular bodies). By contrast, on the 1705 analysis, that indifference constitutes the indeterminacy characteristic of continua but impossible for actual

[23] Leibniz's view of the way in which substances "are in" (*insunt*) bodies is of course considerably less crude than that; for a good discussion of the *inesse* relation between substances and bodies see Rutherford 1990, and see also chap. 2 of Levey 1997.

material objects. Sometime between February of 1676 and 1705, matter and continuity are put drastically asunder in Leibniz's metaphysics, the one being confined to the natural world, the other to the sphere of pure reason. Also by way of contrast, whereas in February points are still candidates for the role of parts, the 1705 account of points drops them to the status of modes, and thus expressly precludes their being parts of anything. Each of these differences of his views of 1705 from those of February 1676 marks a significant change of mind on what are for Leibniz the most central questions of metaphysics. And in each case, the change of mind is evident by November 1676, when he completes his *Pacidius Philalethei*. In these early writings on the continuum, the later metaphysics not only finds its seeds, but has positively taken root.

4. Mathematics and the Actually Infinite Division of Matter

The Cartesian demonstration, if sound, shows that matter circulating through irregular spaces in a plenum must be divided into an actual infinity of parts. Leibniz accepts the key premises of the demonstration—that matter forms a plenum, and that if some portion of matter is to travel through irregular spaces, it must divide into an actual infinity of parts—and dwells upon how to interpret its conclusion. As we have seen, in his initial effort to come to terms with the idea of the actually infinite division of matter, Leibniz pictures matter as "nothing but a multitude of points." But now that picture cannot suffice, for issuing from the new analysis of the continuum is a metaphysics of points according to which they are termini, but never parts, of *extensa*. That metaphysics brings Leibniz to the conclusion that nothing, including matter, could be either resolved into or composed out of points, and it is the first of two factors that lead him to abandon his initial account of the composition of matter.

The second factor is his discovery of an alternative to the initial account of the composition of matter: a model of how a finite portion of matter might be divided into an actual infinity of parts *without* being resolved into a "powder of points." The discovery occurs in the course of his mathematical studies, specifically, in his study of infinite numerical series. He comes to see the possibility of an analogy between an infinite partition of a portion of matter into parts and infinite convergent series of numbers.[24] For example, the infinite series of descending fractions known as "Zeno's series"—½, ¼, ⅛, $\frac{1}{16}$, $\frac{1}{32}$, etc. *ad infinitum*—has a finite sum (it is equal to 1), and appears to contain only smaller and smaller finite fractions at every place. Taking the terms in the series as analogous to the elements of a partition, the descending series of fractions seems to provide an abstract structure that consistently models the conditions imposed on the account of the composition of matter by the Cartesian account of motion and Leibniz's new metaphysics of points. The infinite convergent series (1) comprises an actual infinity of elements, (2) each of which is finite and divisible (that is, non-minimal), and (3) the series has only a finite total combined magnitude.

The proposed analogy between the structure of the infinite convergent numerical series and the structure of material reality, if it holds, should help greatly to show how it is possible for matter to be divided into parts in the way Leibniz claims it actually is. If his metaphysics of points as termini but not parts of things closes off one imagined position in the space of theories of the composition of matter, the study of infinite numerical series guides Leibniz toward another, precisely by bringing into the open a previously unseen possibility. Below, in section 8, I shall sketch out part of the Leibnizian picture of material reality that issues from that newly revealed possibility. But there is in fact a significant obstacle to be overcome before the material end of the proposed analogy can be spelled out. Leibniz has first to clarify and defend its *mathematical* end. For it is not only the analogy that Leibniz discovers and secures in 1676, it is the modern concept of the infinite convergent series itself.

5. *Arithmetica Infinitorum* and the Mathematical Conception of Infinite Series

Securing the concept of the infinite convergent numerical series is no small task. The concept of the infinite convergent series as it

[24]Benardete (1964) seems to be the first to have realized that Leibniz sees an analogy between infinite convergent series of numbers and the division of an *extensum* into an infinity of parts, based on a 1698 letter of Leibniz's to Bernoulli, which he cites (19); Arthur (1989) adroitly picks up this lead (see 187, 199 n. 17). Neither of those commentaries, however, observes the source of the analogy in Leibniz's early writings or its role in prompting his subsequent reevaluation of mathematics.

occurs in the most powerful mathematics available at the *start* of 1676—*Leibniz's* mathematics—appears to attribute to such series properties that would make them unsuitable to serve as models for the infinite division of matter Leibniz envisions. In particular, the technique used for finding sums of infinite numerical series implies that such series would always contain a *terminal* element bounding the diminishing end of the series. This element would be the *infinitieth* term in the series and would be of infinitely small magnitude. We've already seen that Leibniz rejects the existence of infinitesimal parts of matter, and soon we shall see that Leibniz cannot allow even the possibility that there should be an infinitieth term in a series—no matter what size that term might have.

If he is to appeal to the concept of an infinite convergent series of numbers as a model for understanding how a finite portion of matter could be divided into an infinity of finite and divisible parts, Leibniz must first revise his own mathematical theory of the infinite convergent numerical series to remove the commitment it carries to the terminal element. This project leads into deep conceptual waters, and Leibniz pursues it well into them. While the most famous elements of his mathematics (such as the "fundamental theorem" and algorithms of the calculus) are in place by the winter of 1675–76, I maintain that the changes that Leibniz introduces in response to the puzzle about the division of matter must be seen as effectively dividing his mathematics of 1676 into earlier and later periods. One period comes before, and one after, the time when Leibniz, with an eye towards the metaphysics of matter, clarifies the concept of the infinite convergent numerical series.

To see how Leibniz clarifies the concept of the infinite convergent numerical series, however, we need to begin with a sketch of a technique fashioned during a phase of his mathematical development still four years earlier on. For the commitment to the existence of an infinitieth term bounding the infinite series is essentially inherited from a mathematical technique he has developed by 1672: the "method of differences."

Leibniz's first major mathematical discovery is a theorem about the summation of the consecutive terms of a series of differences.[25] Consider a finite series of terms, $a_1, a_2, \ldots, a_n, a_{n+1}$. We can con-

[25]See Hofmann 1974 for the classic account of Leibniz's earliest mathematical development (esp. 14–16); cf. Mancosu 1996 (154–55).

struct another series b of terms whose elements are the *differences* of the consecutive terms of the original a-series:[26]

$$b_1 = a_1 - a_2,\ b_2 = a_2 - a_3, \ldots\ b_n = a_n - a_{n+1}.$$

Leibniz discovers that the sum of the consecutive terms in the b-series is equal to the difference of the extreme terms in the a-series:

$$b_1 + b_2 + \ldots + b_n = a_1 - a_{n+1}.$$

That is, the sum of the consecutive terms of a "difference-series" is equal to the difference of the extreme terms in the original or "base" series. As a concrete example, consider this series of squares and its difference series:

base series a:	0		1		4		9		16		25
difference series b:		1		3		5		7		9	

We thus find that the sum of the first n odd numbers (1 + 3 + 5 + 7 + 9) is equal to the difference of the 1st and (n+1)th squares ($|0 - 25| = 25$).

As Hofmann aptly notes here, "considerations of this sort led [Leibniz] to the conviction that we should be able to derive the sum of any series whose terms are formed by some rule, even where one has to deal with infinitely many terms—assuming only that the expected total sum approaches some finite limit" (1974, 14). We'll see in a moment that Leibniz essentially *extrapolates* his technique for finding sums of arbitrarily large finite series in order to compute the sums of infinite numerical series. More exactly, he extrapolates it to find sums of infinite *convergent* numerical series, the only sorts of series whose sums approach some finite limit.

Before we proceed to the example, however, notice how Leibniz's technique for finding the sum of a difference series relies expressly on the fact that the original series has a *last* term as well as a first one. It is the difference of the first and last term, or the "extremes," of the base series that gives the sum of the terms in the difference series. When these properties of finite series are

[26]It needs to be understood throughout that differences are *absolute* values: $m - n = n - m$. Except in a few cases, the formulae in the text above will leave the notation for absolute value (and parenthesis) implicit.

extrapolated to infinite convergent series, the supposition that the base series contains a final element will remain in force.

Let's now consider a simple problem as an example of this, a problem that Huygens posed to Leibniz: to find the sum of the infinite series s of fractions $^2/_{t(t+1)}$ for natural numbers $t > 0$. That is, to find the sum:

$^1/_1 + {}^1/_3 + {}^1/_6 + {}^1/_{10} + {}^1/_{15} +$ etc.

In order to apply Leibniz's technique there must be some series a of terms for which the terms s_i of series s are the differences of a's consecutive terms, that is, $s_n = a_n - a_{n+1}$. (Here the terms s_i of series s make up the "extrapolated-to-the-infinite" form of what we just discussed above as "the b-series.") And Leibniz notices that, indeed, each term of s,

$^2/_{t(t+1)}$,

can be rewritten as a difference,

$^2/_t - {}^2/_{t+1}$.

That is, the series a is the series of fractions $a_1 = {}^2/_1$, $a_2 = {}^2/_2$, $a_3 = {}^2/_3$, ... $a_n = {}^2/_n$; and the series s is a's difference series, $s_1 = (a_1 - a_2) = {}^2/_1 - {}^2/_2$, $s_2 = {}^2/_2 - {}^2/_3$, ... $s_n = {}^2/_n - {}^2/_{n+1}$, etc. Series a is the base series relative to which s is a difference series. So, in order to find the sum of the terms of series s, we simply take the difference of the extreme terms in a: the sum of the s-terms is equal to $a_1 - a_{n+1}$, that is, $^2/_1 - {}^2/_{n+1}$. But we are supposing here that series s, *and* thus also series a, is taken in its complete, infinite extent. Thus the "extreme terms" are now supposed to be the extremes of an infinite series. The outer extreme term, $^2/_{n+1}$, is therefore supposed to be the *last* term in the series, the term that bounds its diminishing end. Hence, we find the difference of a's extreme terms to be

$^2/_1 - {}^2/_{n+1}$,

where the value of n is now infinitely large. So, the value $^2/_{n+1}$ is infinitely small, or effectively null, and the difference of the extreme terms $({}^2/_1 - {}^2/_{n+1})$ is simply taken as $^2/_1 = 2$. And thus the sum of the infinite series $^2/_{t(t+1)}$, for $t > 0$, is equal to 2.

Now, at last, the postulated terminal element of the infinite series stands before us in high relief. In extrapolating to the infinite, the Leibnizian method of differences used to calculate the sum of the series $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{16}$, etc. *ad infinitum*, is seen to involve pretty unmistakable quantification over a last element that bounds the series at its diminishing end. Several perplexities arise here, perhaps the most obvious of which is the fact that the postulated last term is itself an infinitesimal magnitude. But in the present setting the apparent commitment to infinitesimals is actually something of a side issue. The focus of our immediate concern lies, rather, on the concepts that are responsible for bringing the infinitesimal into play in the first place. The point of particular importance is that the concept of the infinite series encoded in this mathematical technique represents the infinite convergent series as both *infinite* and, in Leibniz's words, *terminatum*, or, as I shall say, *bounded*.[27] Those two predicates make a fairly surprising pair, as 'the infinite' ordinarily suggests just what has no bounds or limits (on at least one "end"). But Leibniz takes pains to distinguish the unbounded (*interminatum*) from the infinite. In this passage he reports that distinction with respect to lines:

> Thus I call 'unbounded' that in which no last point can be taken, save on one side. But by 'infinite' I understand a quantity, either bounded or unbounded, greater than any that can be assigned by us or that can be designated by numbers. (1676. *De Quadratura Arithmetica*, 133)

Mutatis mutandis, his remarks extend to other infinite quantities as well, and most easily to infinite numerical series: an unbounded series has no extreme term on at least one end; an infinite series has more members than can be assigned or designated by num-

[27]There is a slight difficulty here concerning terminology and translation. I render Leibniz's *terminatum* as 'bounded' and *interminatum* as 'unbounded', relying on a roughly vernacular sense of those English terms. ('Terminated' and 'unterminated' aren't satisfactory as *English* renderings.) This choice, however, risks confusion with various technical uses of 'bounded' and 'unbounded' in mathematics (as would 'limited' and 'unlimited'). On one nearby use of 'bounded', a series of terms may be "bounded by" a term not in the series. The series itself has no last term, but each term in the series is (say) less than, and thus "bounded by," a given term; for example, the real numbers in the interval [0,1] have no last member but have the number 1 as a (least upper) bound. Recalling a suggestion in note 19 above, we might understand Leibniz's *terminatum* as '*intrinsically* bounded'. But as 'bounded' is to track Leibniz's (unqualified) *terminatum*, I shall say simply that a series is "bounded by" its terminal elements, and leave the qualifier 'intrinsically' implicit.

bers. Leibniz is also quite clear that a quantity is fit as an object for possible "measurement," such as addition, summation, or multiplication, *only if* it is bounded. So any infinite quantity that can be "measured" in his mathematics must likewise be conceived as a bounded quantity. Thus the infinite convergent series, which is after all a measurable quantity, is conceived to be a bounded infinite. I shall call this conception of the infinite convergent series as bounded and infinite—that is, the conception apparently encoded in Leibniz's actual mathematics—the "mathematical conception" of the infinite convergent series.

By construing the infinite convergent series as infinite and bounded, the mathematical conception of those series runs into some conceptual difficulty, and now we shall come quickly to the heart of the matter. An infinite convergent series is, like the series of natural numbers, a well-ordered series of consecutive terms: roughly, its terms form a single line beginning with a definite first element, each term is immediately neighbored (preceded or succeeded) by at least one other, and any two terms in the series are connected by a "chain" of immediately neighboring terms. Staying with our previous example of the series $1/1$, $1/3$, $1/6$, $1/10$, $1/15$, etc. *ad infinitum*: $1/1$ is immediately succeeded by $1/3$; $1/3$ is immediately preceded by $1/1$ and succeeded by $1/6$; $1/6$ is preceded by $1/3$ and succeeded by $1/10$, and so on. Intuitively, this ordering should allow us to assign a distinct *ordinal number* to each term in the series in a natural way, according to the position it occupies: $1/1$ is the *first*, $1/3$ is the *second*, $1/6$ is the *third*, $1/10$ is the *fourth*, etc. (The finite ordinals simply coincide with the natural numbers: 1, 2, 3, 4, etc.) But now consider the postulated last term. What ordinal assigns its place in the series?

Leibniz (like his early modern contemporaries) tends to speak of this element as the series's *infinitieth* term. This infinitieth term is apparently pictured as occupying a terminal position that is located within the series yet somehow infinitely distant from the first term. Indeed, the infinitieth term appears to be infinitely distant from every term in the series whose position is assigned by a finite ordinal number: it is infinitely distant from the first, infinitely distant from the second, infinitely distant from the third, etc. In fact, any element of the series that is only finitely distant from *one* end of the series must necessarily be infinitely distant from the *other* end.

But recall that any two terms in the series are supposed to be connected by a chain of consecutive terms. If we pick two terms that belong to (that is, are finitely distant from) opposite ends of the series, the chain that connects them must be made up of infinitely many terms. The very idea of such a chain, however, grows bewildering upon examination. As its terms are consecutive, it seems that one should be able to pass from one term on the chain to any other by some familiar step-by-step procedure, such as counting off the intervening terms by ones, or twos, or tens. But as its two ends are infinitely remote from one another, no such procedure could successfully carry one across the chain from a term that belongs to one end of the series to a term that belongs to the other end. In particular, no way of counting off the successive terms beginning with the first (or any "early" term) will allow one to reach the infinitieth term. The idea of the "bounded infinite" chain is thus enshrouded in mystery. Yet that idea is the very essence of the mathematical conception of the infinite series; hence, the mysteriousness of the bounded infinite chain attaches equally to the mathematical conception itself.

There are two diagnoses of the mystery of the bounded infinite chain worth mentioning. The first is defended by, for example, Descartes, who locates the trouble of traversing from one end of the chain to the other as residing merely in his inability to carry out the counting procedure:

> I notice that, when I count, I cannot reach a largest number, and hence I recognize that there is something in the process of counting that exceeds my powers. (AT 7:139)

Descartes, however, is quite mistaken. His powers are more than adequate for the task of counting. To see what goes wrong in his diagnosis, we must turn to the correct one, the second on our menu, which was famously unearthed during the late-nineteenth-century work of Frege and Cantor on the foundations of mathematics. Details aside, the vital point is that no "infinitieth term" lies *within* the series at all. While it makes sense to speak of an infinite ordinal number, no such number is connected by a chain of consecutive terms to any of the finite ordinals—that is, to any of the natural numbers.[28] Restricting our attention to the first in-

[28]There is of course a perfect parallel here with the cardinal numbers: while there is such a thing as an infinite cardinal, no chain of consecutive terms connects it with any of the finite cardinals—that is, with any of the natural numbers.

finite ordinal number, conventionally written ω, we say that while it is true that no ordinal falls between ω and the natural numbers (or finite ordinals), ω nonetheless falls outside the series of natural numbers. With no intervening chain of consecutive terms to connect them, it is impossible to pass from any finite number to ω by counting. *Pace* Descartes, nothing in counting "exceeds one's powers"; rather, no process of counting could possibly traverse from the finite to the infinite ordinal ω. Not even God could "count to infinity."

To adopt Frege's and Cantor's diagnosis, however, involves rejecting the mathematical conception of the infinite series, and moreover, rejecting a particular set of principles that it codifies. I shall argue below that Leibniz does precisely that, singling out and rejecting a mathematical principle that runs deep in early modern thought, and in effect anticipates some of the revolutionary insights that will, two hundred years later, spark the most profound developments in mathematics. At present, however, I want to point out how Leibniz is already committed to rejecting the mathematical conception of the infinite series, quite independently of any revolutionary insights.

He argues patiently, against the mathematical conception, that an infinite series of terms should not be supposed to have a terminal, infinitieth term. For the number needed to designate the place of that infinitieth term in the series would be the "last number"—what we might for now call the number 'infinity'[29] and suppose to be the greatest of all numbers—and, he contends, it is impossible that there should be such a number.

> If there are ten terms, then there is a tenth; but it is debatable whether it follows from this that, if there are infinitely many terms, then there is an infinitieth one. Someone might say that an inference from the

[29] It may be tempting here to use the modern apparatus of ordinals and take this infinite number to be the infinite ordinal number ω. But that would be misleading. For as it is defined by the actual theory of ordinals, ω cannot possibly represent the position in a series that the mathematical conception ascribes to the supposed infinitieth term, namely, the far end of a consecutively ordered, "bounded infinite" series. We might say that the mathematical conception of the infinite series tacitly embraces a *fantasy* theory of ordinals, according to which ω is (*per impossibile*) located within a consecutively ordered, bounded infinite series. Part of Leibniz's rejection of the mathematical conception will involve realizing that this fantasy theory is in fact false.

> finite to the infinite is invalid in this case. . . . It could equally well be argued that, since among any ten terms there is a last number, which is also the greatest of those numbers, it follows that among all numbers there is a last number, which is also the greatest of all numbers. But I think that such number implies a contradiction. (Letter to Bernoulli, 1699. GM 3:566)[30]

Leibniz actually has two arguments for the claim that the concept of a "last" or "greatest" number "implies a contradiction." The first is that no number can be the greatest because every number has a double, and a number's double is always greater than it, thus contradicting the original hypothesis. By current lights, this argument can be seen implicitly to presuppose that there are no infinite numbers; for by the standard according to which $2n > n$, n must be a finite number. Although Leibniz was in no position to know it, that standard is false for infinite n.[31] Thus, his first argument is not cogent, as it fails to show that no infinite number—and, hence, that no number at all—could be "the greatest of all numbers." His second argument, however, is intended directly to refute the hypothesis of an infinite number. (We might, on Leibniz's behalf, consider the two arguments in combination, the first predicated on the success of the second thus: if there are no infinite numbers, then by the standard $2n > n$, there is also no greatest number n.). The style of that second argument is going to be a familiar one, for it reenacts the argument Leibniz offers against the claim that the continuum could be composed of infinitely many points: the hypothesis of the existence of an *infinite number,* he holds, would violate the part-whole axiom and thus would engender Galileo's paradox—a result supposed fatal to any proposition that entails it.

Now let's see how the hypothesis of an infinite number is supposed to have that fatal entailment. Numbers (other than one and zero) are construed by Leibniz to be aggregates of "unities" or "ones," for example, $6 = 1+1+1+1+1+1$ (A 6.3:518)—indeed, the concept of number is essentially *defined* by him in that way[32]—

---

[30]This passage is from a 1699 letter to Bernoulli; but it neatly reflects Leibniz's 1676 thought on the matter. (As with many such arguments he offers to Bernoulli, this one is a well-oiled product originally fashioned in his 1675–76 Paris writings on mathematics.)

[31]I cannot go into the details of this claim here; see Cantor's *Contributions to the Founding of the Theory of Transfinite Numbers* (1955), or chapters 8 and 9 of Russell 1919.

[32]For instance, at A 6.2:441 he writes, "I define number as one and one and one, etc., ôr as unities." For a good example of Leibniz's effort to

and numbers are viewed as applying to aggregates of things taken as a whole rather than to uncollected things taken as so many individuals. Where the number one or "unity" applies to an individual thing, the number five applies to an aggregate "whole" with five "parts" (what we would likely consider to be a *set* with five *members*). So on his view, the concept of an infinite number is bound up with the concept of an infinite aggregate or whole in two ways. First, an infinite number itself would count as an infinite whole; and second, such a number would apply to infinite aggregates or wholes. Consequently, the concept of an infinite number is also doubly bound up with Galileo's paradox: both the infinite number and what it *numbers* would violate the part-whole axiom, if either were to exist.

But as before, Leibniz's *reductio ad paradoxum Galilei* isn't convincing as a refutation: the criterion according to which the part fails to be "less than" the whole—namely, that the part fails to have a lesser number of elements than the whole—is not obviously the same criterion for "less than" that makes the part-whole axiom true. For now, however, we need only to note that Leibniz takes Galileo's paradox to refute decisively the hypothesis that there could be an infinite whole, and thus, on his account of number, to refute the hypothesis that there could be an infinite number as well.

At this point we should take stock. I have argued that Leibniz's actual mathematics for calculating the sums of infinite series encodes a certain conception of the infinite convergent series, what I call "the mathematical conception," according to which there is a terminal element bounding the series at its diminishing end. That terminal element is supposed to be the infinitieth term in the series, and is itself an infinitesimal magnitude. We knew already that Leibniz denies the existence of infinitesimals. We now see also that he denies the existence of an infinite number. But the mathematical conception evidently implies the existence of both. Thus, we see more than just a puzzle about how to make sense of his views of the composition of matter: Leibniz's philosophical views

demonstrate a proposition of arithmetic with number so defined, see *New Essays* 4.7.10 (A 6.6:413f.); and see Frege's beautiful criticism of that demonstration ("we can discover a gap in the proof") in §6 of *The Foundations of Arithmetic* (1950, 7f.)

about mathematics—his denials of infinite and infinitesimal numbers—appear to be in conflict with the commitments of his own mathematical practices. Something has to give if coherence is to be maintained. In "Infinite Numbers," a cluster of notes dated to 10 April 1676, Leibniz confronts these difficulties and submits the concept of the infinite convergent series and its use in mathematics to close scrutiny. From this renewed inquiry into mathematics emerges a stable and subtle philosophy of mathematics and philosophy of the infinite, and in the process the passage to the later period of his 1676 mathematics is transacted.

6. Philosophy of Mathematics in "Infinite Numbers"

That his actual mathematical practice trades on the idea that an infinite convergent series is a bounded infinite counts for Leibniz somewhat as evidence in favor of the correctness of the mathematical conception of those series, but it hardly closes the issue concerning the philosophical significance of that practice. In particular, it remains open to him to assume an *operationalist* stance with respect to those portions of mathematics that employ the bounded infinite. That is, it is possible for him to hold those statements of mathematics whose demonstrations rely essentially on the concept of the bounded infinite to be practically useful but not rigorously true. And in fact he is quick to pursue exactly this option.

He considers the statement that the ratio of the areas of a circle and of a square whose side is the circle's diameter is the same as the ratio of 1 to the sum of the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} +$ etc. (known today as "Leibniz's series"). The purported sum of this series is equal to $\pi/4$, and it is known in advance that the area of the square is to that of the circle as 1 is to $\pi/4$. The worry focuses tightly on the alethic status of the statement that explicitly involves the measured infinite series. Is it true, strictly speaking, that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} = \pi/4$? What is in question is whether the use of the sum of an infinite series to give one of the values in the equation is fully legitimate, or whether instead, on the highest standards of rigor, that use disqualifies the equation's claim to truth. Leibniz concludes the latter.

> We must still investigate whether, and to what extent, the following is true, namely that the square is to the circle as 1 to $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

> + $1/9$ − $1/11$ + etc. For when we say 'etc.' or 'to infinity', the last number is not really understood to be the greatest number, for there isn't one, but it is still understood to be infinite.[33] But as the series is not bounded, how can this be the case? For something must be added, even if it is assumed to be an infinite number, so that it must be said that this (equation) is not rigorously true. (A 6.3:502)

Leibniz here places the postulated last term bounding the infinite series on center stage; he is isolating the source of the conceptual difficulty in his mathematics. That last term is "understood to be infinite," that is, it is understood to be the infinitieth term in the series. Leibniz could easily opt to refute the hypothesis of such a last term with the argument sketched earlier, that the existence of the last term in an infinite series implies the existence of an infinite number, which Galileo's paradox shows is impossible. But his aim here is conceptual clarity, rather than a routine *reductio* of the mathematical conception, and he puts his finger on the crux of the issue: it simply is not true that the infinite convergent series is *terminatum.*

In *De Quadratura Arithmetica* (1676) Leibniz says expressly that we engage in a certain "fiction" when, in order to find the value of an infinite quantity, we suppose that it is bounded (133), as in the example of the last passage "something must be added" to the infinite series in order to calculate its sum. That "something" does not in fact occur in the series. The mathematical conception of the infinite series smuggles in a fictional terminus under its interpretation of the rider 'etc.' or 'to infinity' that is inevitably affixed to the expression of the series as "measured" or as having a sum. Since that series does not in fact contain a last term, the proposition involving its measured form engages directly (if tacitly) in a fiction, and so must be said not to be rigorously true.

If talk of the sums of infinite series is not to be taken as rigorously true, it may nonetheless be read operationally, as a dispensable shorthand for some other, strictly acceptable mathematical

[33] Leibniz is slipping here from discussing the last term in the series to discussing its ordinal number. His claim that the last number "is still understood to be infinite" must concern the last term's *ordinal position* in the series rather than its *magnitude,* since obviously the magnitude of the terms in the series approaches zero, or at least the infinitesimal, as their place in the series approaches infinity, and thus the last term would not be infinitely great.

claim, so that it avoids the fiction of the bounded infinite without an appreciable loss of its mathematical power. This becomes Leibniz's strategy, and his operational reading of that talk is, characteristically, a shining example of theoretical elegance.

> Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like. (A 6.3:503)

Talk of the sum of an infinite series is thus cashed in for rigorously true generalized talk of the sums of finite series and of the rules for producing them. And moreover, Leibniz's style of paraphrase perfectly anticipates Cauchy's definition of the sum of an infinite convergent series as the limit of its partial sums. On that definition, the sum of the convergent infinite series $t_1, t_2, t_3, \ldots t_n, \ldots$ is calculated as follows: Its partial sums, $a_1 = (t_1+t_2)$, $a_2 = (a_1+t_3), \ldots$ $a_n = (a_{n-1} + t_{n+1}), \ldots$ form a further series $a_1, a_2, a_3, \ldots, a_n, \ldots$ Now to define what it is for the series a_n to converge to the limit a as $n \rightarrow \infty$ ("goes to infinity"). The idea here is that the terms of the series successively approach closer and closer to the limit a without ever reaching or surpassing it, and moreover the *difference* between a and successively later terms in the series shrinks towards zero without ever reaching or surpassing it, while passing any positive rational number you might mention, however small, along the way. "The error [the difference from the limiting value] always diminishes as the series increases, so that it becomes as small as we would like." Infinitely small? No, and this really is the point, just smaller than any positive finite value one might choose.[34]

With the operationalist reading in hand, Leibniz is able consistently to make sense of talk of the sums of infinite numerical series

[34]Compare Cauchy's apparatus of the epsilon-concept. We state that the series $a_1, a_2, a_3, \ldots, a_n, \ldots$ converges to the limiting value a if and only if, for however small a positive rational number e you specify, you can always find a neighborhood of terms surrounding a, constituting an interval of length e, that contains *more* elements of the sequence than lie outside it. Only finitely many elements of the series a_n could lie outside any such neighborhood of a. We can now say that the limit of the series a_n, as $n \rightarrow \infty$, is a if and only if $(a - a_n)$ is less than e for all n sufficiently large: $n > N$. See Benardete 1964 (11–14) for a nice discussion of the theory of infinite convergent numerical series and its ontological implications.

without having to embrace the "bounded infinite." The operationalist account of the sums of infinite numerical series has no need to postulate a last term in an infinite series, and thus it has to admit neither the existence of the mythical infinitieth term, nor that of infinitely great or infinitely small numbers. Once the mathematical conception of the infinite series is replaced with the operationalist one, it becomes possible for all those philosophically undesirable commitments to be severed from his mathematics. *Possible.* But actually cutting the conceptual links that are imagined to connect infinite series of numbers to infinitary numbers requires further clarifying the mathematical facts, modifying the philosophy associated with them, and negotiating two final difficulties. Indeed, the compass of his project of clarifying the concept of the infinite series quickly extends to the very foundations of mathematics.

7. Philosophy of the Infinite in "Infinite Numbers'

Leibniz needs to revise some basic mathematical notions in order to make them cohere with his new understanding of infinite series, and unsurprisingly everything turns on how he deals with the concept of infinity, the concept of number, and the relation between them. It is without delay that he takes up those concepts, first of all distinguishing two senses in which numbers might be said to "go to infinity."

> [N]umbers themselves absolutely *per se* do not go to infinity, since then there would be a greatest number [which would be an infinite number]. But they do go to infinity when applied to a certain space, ôr to an unbounded line divided into parts. (A 6.3:503)

Numbers do *not* "go to infinity" in the sense that there is no infinitieth number of infinitely great magnitude eventually reached within the series. But they do "go to infinity" in the sense that, to each element of an infinity of things, a distinct number can be assigned; and hence, there is an infinity of numbers. Leibniz's position here—that there can be *an infinity* of numbers without there being an *infinite number*—is a delicate one, and it relies heavily on the coherence of the idea of an unbounded (*interminatum*) infinite series. The example of "an unbounded line divided into parts" is supposed to be that of an infinitely long line "in which no last point can be taken, save on one side," which itself is divided into consecutive segments of a fixed unit length, say, a foot or an inch.

The sequence of line-segments that is thus constructed is an unbounded infinite series. The numerical series constructed by "applying" consecutive numbers to the parts of the line is likewise an unbounded infinite series: since there is no last part of the line, there is no last term in the constructed series of numbers. Is the concept of the unbounded infinite series coherent? In the dense and dialectical few lines that follow, Leibniz tests the coherence of his case.

> Now here there is a new difficulty. Is the last number of a series of this kind the last one that would be ascribed to the divisions of an unbounded line? It is not, otherwise there would also be a last in the unbounded series. Yet there does seem to be (a last number), because the number of terms of the series will be the last number. Suppose to the point of division we ascribe a number always greater by unity than the preceding one, then of course the number of terms will be the last number of the series. But in fact there is no last number of the series, since it is unbounded. (A 6.3:503)

The statement that "the number of terms in the series will be the last number of the series" expresses an important, naive mathematical principle, and in that principle two concepts of number may be discerned. First, there is the concept of "the number of terms in the series" or what we might call the *cardinality* of the series. Second, for any given term within the series, there is a number that designates its place in the series: again, this is the concept of the ordinal number of that term. The naive principle links the two number concepts together by insisting that the cardinality of a series is assigned by the ordinal number of a term occurring within the series, namely, the ordinal number of its last term. I said earlier that in rejecting the mathematical conception of the infinite series, Leibniz would single out and reject an important set of mathematical principles integral to the early modern conception of mathematics. The naive principle is foremost among them.

The naive principle threatens to press Leibniz's hypothesis of an unbounded infinite series into a contradiction, because the unbounded infinite series apparently has a cardinality—an infinite cardinality—and yet has no last term. This is the "new difficulty" he mentions, and in responding to it Leibniz either has to modify the hypothesis of an unbounded infinite series so that it does not present a counterexample to the naive principle, or has to reject the principle itself. In fact, he does both.

In his rejection of the naive principle, Leibniz quite simply pushes past the early modern frontiers of mathematical understanding of the infinite:

> Thus if you say that in an unbounded series[35] there exists no last finite number that can be written in, although there can exist an infinite one: I reply, not even this can exist, if there is no last number. To this reasoning I have nothing else to reply, except that the number of terms is not always the last number of the series. That is, it is clear that even if finite numbers are increased to infinity, they never . . . reach infinity. This consideration is extremely subtle. (A 6.3:504)

In seeing his way clear to the fact that the number of terms in the series of natural numbers—the cardinality of the naturals—is not itself in the series, but rather lies outside it, Leibniz places himself well ahead of the majority of his peers and predecessors on the topic. Further, taken at face value, his claim that "the number of terms is not always the last number of the series" touches quite directly on the concept of cardinality, and conceives of a series's cardinality as a *number.* In the crucial case of the infinite series, the number of terms is the *cardinal number* "infinity" (waiving Cantor's distinctions between higher and lower transfinite cardinals), despite the fact that there is no corresponding infinitieth element in the series. Were it stable, this insight would place Leibniz closer still to the nineteenth-century work on the foundations of mathematics.

But in fact the insight is not stable, for without a general analysis of the concept of number and the concept of cardinality, Leibniz cannot find a sound defense for the concept of an infinite cardinal against the problems apparently besetting it. He sees that the concept of "the number of terms in the series," if applied to an infinite series, would yield an infinite number as its cardinality. And again, the concept of an infinite number gives rise to Galileo's paradox, which Leibniz takes to be fatal. The coming revolution in mathematics will have to remain on the horizon, waiting for Frege and Cantor.

Although the concept of the infinite cardinal does not take hold in Leibniz's mathematical philosophy, his brush with it illuminates a last difficulty right at the heart of the concept of the infinite

[35] Here the Akademie editors corrected 'series' to 'line'; I follow Richard Arthur in reversing their correction.

series. For with the idea of a single number associated with the whole of a series of terms now more clearly in hand, Galileo's paradox is seen to arise directly from the infinitude of the series itself, quite independently of the further postulation of a final, infinitieth term in the series. It appears to be exactly as problematic for a series to have an infinite cardinality as it is for it to possess an infinitieth term. Both conditions require the existence an infinite number, the very concept of which involves the "contradiction" that the part is not less than the whole. Note also that the impossibility connected with infinite cardinality is not confined simply to the hypothesis of a *number* that violates the part-whole axiom. Cardinality, as we're treating it, is a property of aggregates of elements, and in the present case it is a property of a series of terms: an infinite *series* ought also to be, *per impossibile*, an infinite whole. Benardete crisply remarks, "It is curious that Leibniz failed to see that this 'contradiction' infects not merely the concept of an infinite number but also the concept of any infinite collection whatever" (1964, 46).

But does Leibniz fail to see this point? To the contrary, he sees it with great clarity and consciously crafts his philosophy of the infinite to avoid any commitment to the infinite collection. To escape Galileo's paradox, both the concept of infinite number and that of the infinite aggregate or whole must be disavowed. Leibniz's strategy unfolds with the idea that infinite number and infinite whole stand and fall together: as before, numbers are themselves aggregates of "unities," and apply to aggregates of things. A commitment to an infinite number arises only if there is, or could be, an infinite aggregate or whole to be quantified by it. So, in order to cast off any commitment to an infinite number, it suffices to bar the existence of any infinite aggregates of things. This cannot take the form of claiming that there are in fact only finitely many things, however, since Leibniz's theory of motion demands that there be an infinity of parts of matter. (Here 'an infinity' is taken as standing in for an infinite *plural* term rather than as a singular term purporting to denote a single entity or "true whole.") The bar against infinite aggregates or wholes has to be a denial that an infinity of things ever compose a whole.

> Therefore we conclude finally that there is no infinite multitude, from which it will follow that there is not an infinity of things, either. Or

> [rather] it must be said that an infinity of things is not one whole, or that there is no aggregate of them. (A 6.3:503)

Thus, Leibniz distinguishes the claim that there are *infinitely many Fs*, or *an infinity of Fs*, from the claim that there is an infinite *aggregate* of those *Fs*. Only the latter, he holds, jeopardizes the part-whole axiom and engenders Galileo's paradox. This involves a flatly metaphysical claim about the unity of what we might neutrally call *a multitude* of elements. Some multitudes, perhaps, form true wholes or aggregates, but an infinite multitude could only be so many separate elements. Composition can occur only over a finite field of elements, and that is to say, strictly speaking, there could not be such things as infinite numerical series or bodies that are actually divided into infinitely many parts. For such things could not be truly unified wholes, and, as Leibniz is eventually to insist, what is not truly *a* being is not truly a *being*, either (G 2:97, 251). What unity an infinite multitude appears to have is not intrinsic to it, but could only be an aspect of the appearance of its infinitely many elements to some mind. In the case of an infinite series of numbers, the appearance to the mind of a unity comes from the mind's grasp of the *rule* for generating the later terms in the series from the earlier ones: the law of the series. Leibniz will eventually come to hold that the comprehension of an infinity as a unity by a mind always follows the model of grasping a law of the series, whether that "unity" is an infinite series of numbers, or, to take another example, an infinitely complex concept of an individual substance (G 2:136, 262f.)

Leibniz's distinction between *an infinity* and *an infinite whole* is to become just the first in a long line of attempts by philosophers of mathematics to escape paradox by distinguishing permissible from impermissible collectings of disparate elements into a unity. For instance, Russell with his "illegitimate classes" and Cantor with his "inconsistent totalities," while confronted with more sophisticated paradoxes, and more subtle in their restrictions on the formation of one out of many, both fall into this lineage, as does the now usual distinction between sets and proper classes.[36]

It remains, however, for Leibniz to explain what it is for there to be "an infinity" of numbers or of things, if it is not for there

[36]See Russell 1908 (63, 75) and Cantor 1967 (114).

to be an infinite number or infinite aggregate of them. His answer is that to say that a series is infinite, or that there is an infinity of things, is not to ascribe a specific number to that series or to those things, but rather is to say that there are more than *n* terms or things, for any (finite) number *n*. Explicit use of the device of plural quantification allows us to construct a more satisfactory rendering of Leibniz's account: there is an infinity of things just in case there are *xs* such that, for any number *n*, the *xs* are more than *n*. I believe that essentially this view is in play already in 1676, although Leibniz tends to slide into expressions of his view that mask the force of the plural reference, as is evident in a passage we've seen once before where he writes, "by 'infinite' I understand *a quantity* . . . greater than any that can be assigned by us or that can be designated by numbers"(*De Quadratura Arithmetica,* 133). Over the following years Leibniz tries out various other ways of stating the view, sometimes striking on formulations that nicely eliminate even the *appearance* of reference to an "infinite whole" in favor of a plural referring expression, as for example in a fragment dated between 1677 and 1683: "Bodies are actually infinite, that is, there exist more bodies than there are unities in any given number" (VE, 1129). In any event, by the late 1690s a preferred formulation has surfaced and remains in stable use thereafter; here it occurs in his correspondence with Bernoulli:

> When it is said that there are infinitely many terms, it is not being said that there is some specific number of them, but that there are more than any specific number. (GM 3:566)[37]

Thus, Leibniz's philosophy of the infinite places the concept of infinity safely outside the scope of the concept of number. The infinite can exist in the realm of mathematics, but not as a single whole; rather, the infinite provides the framework for mathematics, so to speak, appearing only as the infinity of mathematical objects but never occurring within mathematics as a unitary, infinite math-

[37]See also the celebrated passage from the 1704 *New Essays*: "It is perfectly correct to say that there is an infinity of things, that is, that there are always more of them than one can specify. But it is easy to demonstrate that there is no infinite number, nor any infinite line or other infinite quantity, if they be taken to be infinite wholes" (A 6.6:157). For an excellent discussion of "infinite whole," "infinite number," and related matters in both Kant and Leibniz, see Bennett 1974 (125–28).

ematical object. Likewise for the world of matter: there can be an actual infinity of matter's parts, but, perhaps ironically, those "parts" can never constitute a true whole.

8. A Metaphysics of Infinitely Divided Matter

At this point, Leibniz's project in clarifying the concepts of mathematics connected with the concept of the infinite series effectively comes to a close, with most of his ensuing discussions of the topic taking place in a didactic rather than exploratory mode. The later 1676 period of his thought about the mathematics of infinite series is, essentially, his mature thought about it. The mathematical conception of the infinite series has been dispatched, along with its cadre of problematic notions: the "bounded infinite," the infinitesimal, the infinite cardinal, and the infinite series itself, if taken as a genuine whole. All are excised from the rigorously correct understanding of mathematics, and are retained only in their appearances, as facets of a practically useful technique towards which an operationalist stance is to be assumed. Having cut the infinite series free from its imagined undesirable consequences, Leibniz can at last secure the concept of the infinite convergent numerical series, an unbounded infinite series consisting entirely of finite terms that can be said to have a finite sum. And this brings us back to his inquiry into the metaphysics of matter, for the very point in clarifying and securing that concept of the infinite series has been to isolate an abstract structure that could serve as a coherent model for his account of the infinite division of matter.

As a consequence of its circulating through irregular spaces in a plenum, a portion of matter would have to be divided into an actual infinity of parts. But these parts could not be "metaphysical points" or minima, since, as the analysis of the continuum revealed, minima can only be termini and never parts of things. Nor is it correct to suppose that those parts are actual infinitesimals. Rather, it must somehow be possible for a finite portion of matter to be divided into an infinity of parts, each of which is divisible and of some finite size. The infinite convergent numerical series makes its metaphysical contribution here by providing precisely such a possibility. A finite portion of matter might divide into an infinity of finite and divisible parts just as a finite number might divide into an infinity of finite and divisible numbers. The moving portion of matter, as it adjusts its shape, divides into an infinity of smaller parts, each of which is surpassed in smallness by some other part but none of which is indivisible or infinitesimal. Again, Zeno's series makes for a nice example: a portion of matter 1 cubic foot in volume travels through an irregular space, and in the process breaks into an infinity of parts. One of those parts is half the size of the original portion of matter, measuring $\frac{1}{2}$ cubic foot in volume. Another part is only a quarter the size of the original, measuring $\frac{1}{4}$ cubic foot. Still another is $\frac{1}{8}$ cubic foot; and yet another is $\frac{1}{16}$ cubic foot. And so on *ad infinitum.* No part is such that no other part is smaller still, and more importantly, no part is minimal and no part is infinitesimal.

For the present Leibnizian model of the structure of matter to satisfy the premises of the Cartesian demonstration that originally provoked Leibniz to explore the idea of infinite division, it must be possible for the infinity of parts (1) to form a plenum and (2) to allow for motion. Presumably, this will be a matter of the parts having just the right shapes and falling into just the right sorts of configurations. Leibniz at least once envisions it as a case of sphere packing:

> if everything is thought to be full of globes, then between the interstices new globes can be placed to infinity without violating motion, for it is only necessary for the smaller globes to move more swiftly. ("Metaphysical Definitions and Reflections," *c.* 1680. VE, 2040)

Although the tone of those lines suggests more confidence than Leibniz is probably warranted to have on the subject, his idea that an infinity of "globes" could be arranged in such a way as to form a plenum and allow for motion may yet be borne out by contemporary work on sphere-packing problems. A typical sphere-packing problem asks how a space of a given shape might be packed with non-overlapping spheres, subject to any of a number of various constraints. (Usually those constraints fix the size of the spheres, say, fixing them all to have a given radius, and the packing problem concerns what the upper limit on the number of spheres packable into the given space is. Obviously Leibniz's packing problem does not have this character.) If two of those constraints are that the spheres fill the space and that motion be possible within it, the difficulty of the problem increases by at least an order of magnitude. It would be absurdly optimistic to suppose that an infinity of globes *conforming to the design of Zeno's series*—one globe that is half the size of the original portion of matter, and each globe being exactly twice the size of one other—would be able to satisfy both.

But there is no need to restrict ourselves to such narrow parameters on the sizes and shapes of the parts into which a portion of matter divides; there are infinitely many distinct infinite convergent series of numbers, and infinitely many distinct shapes, that might be employed in modeling the division of matter into an actual infinity of parts. If it is implausible that matter could divide into a prettily descending infinite sequence of globes and still satisfy the constraints of the packing problem,[38] an infinitely more motley assortment of ever smaller parts might yet do. For the purpose of fleshing out a Leibnizian metaphysic of infinitely divided matter, sphere packing can give way to packings of other objects of a variety of less familiar shapes. Though the chances that anyone could actually calculate an answer to such an object-packing problem may turn out to be vanishingly small (and it is certainly no part of this paper to chase such an answer), still, the sort of packing called for has not been shown to be impossible. And I think it would not be unreasonable to hope that somewhere in the range of possibilities there exist configurations of an infinity of finite and divisible parts of matter that satisfy the two constraints placed on that problem. That would be a large step in the direction of success for Leibniz's project.

The infinite series model of the infinite division of matter of course has an added price for Leibniz, a price exacted by Galileo's paradox. The portion of matter that resolves into an infinity of parts cannot remain a unity during the interval of its motion

[38] In fact, it is now known (though it was not in Leibniz's time) that no series of globes can fill a space *perfectly*, if we understand that requirement in the following strong sense: every point in the space has to lie either on the surface or in the interior of a globe. The most efficient sphere packings still leave a remainder of unfilled points in the space; remarkably, the remainder is essentially Cantor's "ternary set": it has the cardinality of the continuum but is nowhere dense. Thus, the "gaps" in the space are isolated single points, and have measure zero. (Mandelbrot (1982, 404) calls this remainder the "Cantor dust.") But it should be noted that by Leibniz's lights, a space with only such a remainder left unfilled would still constitute what he calls a "physical plenum," and it is this less demanding sort of plenum that he in fact ascribes to nature (A 6.3:473). For some discussion, see my "Discontinuity and the Structure of Motion in Leibniz," in progress.

through the irregular spaces; otherwise it would, *per impossible,* be an infinite whole. At most a finite number of those parts can compose a single material body at once, so at least some of the "parts of matter" actually divided off by their motion—at least infinitely many of them, in fact—must lose their status as parts of the moving whole. To be sure, there is no contradiction in that result, but it must certainly complicate the story Leibniz has to tell about the nature of motion. Taken together, the constraint on composition that rules out infinite wholes and the constraint on "adjustments of shape" that requires infinite division into parts generate a fascinating set of scenarios concerning the unity of a moving object. But it would be misleading to explore them in the present paper, as Leibniz's theories of motion and cohesion add to the equation several unusual variables that I cannot properly address here.[39] It is worth noting, however, that those variables are independent of the elements of Leibniz's metaphysics of matter that we have detailed thus far. To the extent that we have recovered that metaphysics (having focused tightly on certain topics to the exclusion of others, ours is very much an incomplete portrait), what we have found is a picture of reality that accomplishes to a remarkably high degree of success what it sets out to do: to show how it is possible that a finite portion of discrete matter might divide into an actual infinity of finite and divisible parts.

The analogy Leibniz discovers in 1676 between infinite convergent series of numbers and the actual infinity of matter's parts becomes a longstanding feature of his metaphysics of matter. More than twenty years later, for instance, he unlimbers it when Bernoulli, who is in the grips of the old mathematical conception of the infinite series and its attendant notions, challenges the coherence of his doctrine of the infinite division of matter. Bernoulli writes,

> You admit that any finite portion of matter is already actually divided up into an infinite number of parts; and yet you deny that any of these parts can be infinitely small. How is this consistent? (GM 3:529)

Leibniz's response elegantly incorporates his mathematical model:

> I do not think it follows from [the infinite division of matter] that there exists any infinitely small portion of matter. Still less do I admit

[39]For discussions of Leibniz on motion and cohesion see Arthur 1989 and 1998, and Garber 1982.

> that it follows that there is any absolutely minimum portion of matter. . . . Let us suppose that in a line its $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., are actually assigned, and that all the terms of this series actually exist. You infer from this that there also exists an infinitieth term. I, on the other hand, think that nothing follows from this other than that there actually exists any assignable finite fraction, however small you please. (GM 3:535–36)

The partition of a finite *extensum*—here the example is of a line, but it can be modified to fit the case of a body—into an infinity of parts can consist entirely of lesser *extensa*, all finite and divisible, arranged in a descending geometrical sequence. Neither infinitesimals nor minimal portions of matter have to be postulated in order to make sense of the hypothesis of an actually infinite division.

In capitalizing on the opportunity passed over by Descartes, Leibniz latches onto the cluster of issues that comes to define the center of his philosophical concerns, and in pursuing those issues he pushes the envelope in both metaphysics and mathematics. His 1676 inquiries yield a spate of new hypotheses—about minima, the continuum, mathematics, the infinite, the composition of matter—hypotheses that are to abide as critical components of his fully matured metaphysics. Yet before concluding this discussion it must be mentioned that those inquiries also yield a spate of new difficulties for his own metaphysics of matter, infecting it right at the foundations. The troubles emanate primarily from two sources—a further thesis about motion and another strand in his thought about the mathematics of infinite division—but lead to a common problematic conclusion: it seems that not only is matter divided into an infinity of parts, but *every part* of matter is further divided into an infinity of parts, without end. But this means (*inter alia*) that no part of matter you might specify is truly *one*—that is, no part of matter you might specify truly *exists*. So unless the world of matter is simply unreal altogether, the reality it contains must issue from something else, something immaterial in the foundations of matter whose unity is not subject to the same difficulty of infinite division. Leibniz's name for that "something" will eventually be *monad*. But that is a topic to be taken up in another paper.[40]

Dartmouth College

[40] See my "Leibniz's Constructivism and Infinitely Folded Matter," forthcoming as Levey 1998.

References

Adams, Robert Merrihew. 1983. "Phenomenalism and Corporeal Substance in Leibniz." *Midwest Studies in Philosophy* 8:217–57.

———. 1985. "Predication, Truth and Transworld Identity in Leibniz." In *How Things Are: Studies in Predication and the History of Philosophy and Science*, ed. J. Bogen and J. E. McGuire, 235–83. Dordrecht: D. Reidel.

———. 1994. *Leibniz: Determinist, Theist, Idealist*. Oxford: Oxford University Press.

Arthur, Richard. 1989. "Russell's Conundrum: On the Relation of Leibniz's Monads to the Continuum." In *An Intimate Relation*, ed. Brown and Mittelstrass, 171–201.

———. 1998. "Cohesion, Division, Harmony: Physical Aspects of Leibniz's Continuum Problem (1671–1686)." Forthcoming in *Perspectives on Science: Historical, Philosophical, Social*.

Benacerraf, Paul, and Hilary Putnam, eds. 1983. *Philosophy of Mathematics: Selected Readings*, 2d ed. Cambridge: Cambridge University Press.

Benardete, José A. 1964. *Infinity: An Essay in Metaphysics*. Oxford: Oxford University Press.

Bennett, Jonathan. 1974. *Kant's Dialectic*. Cambridge: Cambridge University Press.

———. 1984. *A Study of Spinoza's "Ethics."* Cambridge: Cambridge University Press.

Boolos, George. 1971. "The Iterative Conception of Set." *Journal of Philosophy* 69:215–32. Reprinted in Benacerraf and Putnam 1983.

Cantor, Georg. 1955. Reprint. *Contributions to the Founding of the Theory of Transfinite Numbers*. Trans. Phillip E. B. Jourdain. New York: Dover. Original edition, Chicago: Open Court, 1915.

———.1967. "Letter to Dedekind." In *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, ed. Jean van Heijenoort. Cambridge: Harvard University Press.

Couturat, Louis. 1901. *La Logique de Leibniz*. Paris: Alcan.

Dedekind, Richard. 1901. *Essays on the Theory of Numbers*. Chicago: Open Court. Reprinted, New York: Dover, 1963.

Descartes, René. 1964–76. *Oeuvres de Descartes*. Rev. ed. Ed. Ch. Adam and P. Tannery. Paris: Vrin/C.N.R.S.

———. 1985. *The Philosophical Writings of Descartes*, vol. 1. Trans. and ed. J. Cottingham, R. Stoothoff, and D. Murdoch. Cambridge: Cambridge University Press.

Donagan, Alan. 1988. *Spinoza*. Chicago: University of Chicago Press.

Earman, John. 1975. "Infinities, Infinitesimals, and Indivisibles: The Leibnizian Labyrinth." *Studia Leibnitiana* 7:236–51.

Euclid. 1956. *Elements*. Ed. T. Heath. New York: Dover.

Frege, Gottlob. 1950. *The Foundations of Arithmetic.* Evanston: Northwestern University Press.

Galilei, Galileo. 1898. *Opere.* Ed. Antonio Favaro. Florence: Edizione Nationale.

Garber, Daniel. 1982. "Motion and Metaphysics in the Young Leibniz." In *Leibniz: Critical and Interpretive Essays,* ed. Hooker, 160–84. Minneapolis: University of Minnesota Press.

———. 1985. "Leibniz and the Foundations of Physics: The Middle Years." In *The Natural Philosophy of Leibniz,* ed. Okruhlik and Brown, 27–130. Dordrecht: D. Reidel.

———. 1992. *Descartes' Metaphysical Physics.* Chicago: Chicago University Press.

———. 1995. "Leibniz: Physics and Philosophy." In Jolley 1995, 270–352.

Grünbaum, Adolph. 1973. *Philosophical Problems of Space and Time.* Dordrecht: D. Reidel.

Hartz, G., and J. Cover. 1987. "Space and Time in the Leibnizian Metaphysic." *Noûs* 22:493–519.

Hofmann, Joseph E. 1974. *Leibniz in Paris, 1672–1676; His Growth to Mathematical Maturity.* Cambridge: Cambridge University Press.

Ishiguro, Hidé. 1990. *Leibniz's Philosophy of Logic and Language.* 2d ed. Cambridge: Cambridge University Press.

Jolley, N., ed. 1995. *The Cambridge Companion to Leibniz.* Cambridge: Cambridge University Press.

Kulstad, Mark. 1991. *Leibniz on Apperception, Consciousness, and Reflection.* Munich: Philosophia Verlag.

Leibniz, G. W. 1849–63. *Mathematatische Schriften von Gottfried Wilhelm Leibniz,* vols. 1–7. Ed. C. I. Gerhardt. Berlin: A. Asher; Halle: H. W. Schmidt.

———. 1875–90. *Die Philosophischen Schriften,* vols. 1–7. Ed. C. I. Gerhardt. Berlin: Weidmannsche Buchhandlung.

———. 1903. *Opuscules et Fragments inédits de Leibniz.* Ed. L. Couturat. Paris: Alcan.

———. 1923–80. *Samtliche Schriften und Briefe. Philosophische Schriften.* Series 6, vols. 1–3. Berlin: Akademie-Verlag.

———. 1969. *Philosophical Papers and Letters.* Ed. Leroy Loemker. Dordrecht: Kluwer Academic Publishers.

———. 1982–92. *Vorausedition.* To *Samtliche Schriften und Briefe.* Series 6, vol. 4. Berlin: Akademie-Verlag. Ten fascicules.

———. 1983. *New Essays on Human Understanding.* Trans. and ed. Peter Remnant and Jonathan Bennett. Cambridge: Cambridge University Press.

———. 1989. *Philosophical Essays.* Trans. and ed. Roger Ariew and Daniel Garber. Indianapolis: Hackett.

———. 1993. *De Quadratura Arithmetica Circuli Ellipseos et Hyperbolae cujus*

Corolarium est Trigonometria sine Tabulis. Critically edited and annotated by Eberhard Knobloch. Göttingen: Vandenhoeck and Ruprecht.

Levey, Samuel. 1997. "Matter, Unity and Infinity in Early Leibniz." Ph.D. diss., Syracuse University.

———. 1998. "Leibniz's Constructivism and Infinitely Folded Matter." Forthcoming in *New Essays on the Rationalists,* ed. R. Gennaro and C. Huenemann. New York: Oxford University Press.

Mancosu, Paolo. 1996. *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century.* Oxford: Oxford University Press.

Mates, Benson. 1986. *The Philosophy of Leibniz: Metaphysics and Language.* New York: Oxford University Press.

McCrae, Robert. 1976. *Leibniz: Perception, Apperception, and Thought.* Toronto: University of Toronto Press.

Mondadori, Fabrizio. 1973. "Reference, Essentialism, and Modality in Leibniz's Metaphysics." *Studia Leibnitiana* 5:73–101.

———. 1975. "Leibniz and the Doctrine of Inter-World Identity." *Studia Leibnitiana* 7:22–57.

Moore, A. W. 1990. *The Infinite.* London: Routledge.

Parsons, Charles. 1977. "What is the Iterative Conception of Set?" *Proceedings of the 5th International Congress of Logic, Methodology and Philosophy of Science 1975,* part 1: *Logic, Foundations of Mathematics, and Computability Theory,* ed. Butts and Hintikka, 335–67. Dordrecht: D. Reidel. Reprinted in Putnam and Benacerraf 1983.

Rescher, Nicholas. 1967. *The Philosophy of Leibniz.* Englewood Cliffs, NJ: Prentice-Hall.

Russell, Bertrand. 1900. *A Critical Exposition of the Philosophy of Leibniz, with an Appendix of Leading Passages.* London: Allen and Unwin.

———. 1908. "Mathematical Logic as based on the Theory of Types." *American Journal of Mathematics.* Reprinted in *Logic and Knowledge: Essays 1901–1950,* ed. Marsh. London: Routledge, 1956.

———. 1919. *Introduction to Mathematical Philosophy.* London: Allen and Unwin.

Rutherford, Donald. 1990. "Leibniz's 'Analysis of the Multitude and Phenomena into Unities and Reality'." *Journal of the History of Philosophy* 28: 525–52.

———. 1995. *Leibniz and the Rational Order of Nature.* Cambridge: Cambridge University Press.

Skyrms, Brian. 1983. "Zeno's Paradox of Measure." In *Physics, Philosophy and Psychoanalysis,* ed. Cohen and Landau, 223–54. Dordrecht: D. Reidel.

Sleigh, Robert C., Jr. 1983. "Leibniz on the Two Great Principles of All Our Reasoning." In *Midwest Studies in Philosophy 8: Contemporary Perspectives on the History of Philosophy,* ed. French, Uehling, and Wettstein, 193–216. Minneapolis: University of Minnesota Press.

———. 1990. *Leibniz and Arnauld: A Commentary on Their Correspondence.* New Haven: Yale University Press.

Sleigh, R., and Christia Mercer. 1995. "Metaphysics: The Early Period to the *Discourse on Metaphysics*." In Jolley 1995, 67–123.

Spinoza, Baruch. 1925. *Spinoza Opera.* 4 vols. Ed. Carl Gebhardt. Heidelberg: Carl Winter.

Wang, Hao. 1974. "The Concept of Set." In *From Mathematics to Philosophy,* 181–223. London: Routledge and Kegan Paul. Reprinted in Benacerraf and Putnam 1983.

Weyl, Hermann. 1949. *Philosophy of Mathematics and Natural Science.* Princeton: Princeton University Press.

Wilson, Catherine. 1989. *Leibniz's Metaphysics: A Historical and Comparative Study.* Princeton: Princeton University Press.

Wilson, Margaret. 1976. "Leibniz's Dynamics and Contingency in Nature." In *Motion and Time, Space and Matter: Interrelations in the History and Philosophy of Science,* ed. Peter Machamer and Robert Turnbull, 264–89. Columbus: Ohio State University Press.