Leibniz: Sum of an Infinite Series

Definition $\{a_j^1\}_j = \{\frac{1}{j}\}_j$. Let $\{a_j^1\}$ be a_j

Here $\{a_j^1\} = \{\frac{1}{j}\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Definition $\{a_j^{i+1}\} = \{a_j^i - a_{j+1}^i\}$

Here $\{a_j^2\} = \{a_j^{1+1}\} = \{a_j^1 - a_{j+1}^1\}$

$= \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots$

Here $\{a_j^3\} = \{a_j^{2+1}\} = \{a_j^2 - a_{j+1}^2\}$.

$= \frac{1}{3}, \frac{1}{12}, \frac{1}{30}, \frac{1}{60}, \ldots$

Let $\{a_j^2\}$ be b_j, $\{a_j^3\}$ be c_j etc.

$$\boxed{\begin{array}{l} b_n = a_n - a_{n+1} \\ a_1 - a_{n+1} = \sum_1^n b_i \end{array}}$$

$$\boxed{\begin{array}{l} b_1 = a_1 - a_2 \\ \frac{1}{2} = 1 - \frac{1}{2} \end{array}}$$

$$\boxed{\begin{array}{l} a_1 - a_4 = \sum_1^3 b_i \\ 1 - \frac{1}{4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12} \end{array}}$$

$$1\,a_1$$

$$\frac{1}{2}a_2 \quad \frac{1}{2}b_1$$

$$\frac{1}{3}a_3 \quad \frac{1}{6}b_2 \quad \frac{1}{3}c_1$$

$$\frac{1}{4}a_4 \quad \frac{1}{12}b_3 \quad \frac{1}{12}c_2 \quad \frac{1}{4}d_1$$

$$\frac{1}{5}a_5 \quad \frac{1}{20}b_4 \quad \frac{1}{30}c_3 \quad \frac{1}{20}d_2 \quad \frac{1}{5}e_1$$

$$\frac{1}{6}a_6 \quad \frac{1}{30}b_5 \quad \frac{1}{60}c_4 \quad \frac{1}{60}d_3 \quad \frac{1}{30}e_2 \quad \frac{1}{6}f_1$$

$$\frac{1}{7}a_7 \quad \frac{1}{42}b_6 \quad \frac{1}{105}c_5 \quad \frac{1}{140}d_4 \quad \frac{1}{105}e_3 \quad \frac{1}{42}f_2 \quad \frac{1}{7}g_1$$

Theorem $\boxed{\sum_{j=1,\ldots,n} a_j^{i+1} = a_1^i - a_{n+1}^i}$, e.g. $\boxed{\sum_1^n b_n = a_1 - a_{n+1}}$

$\boxed{a_1^i = \sum_{j=1,\ldots,n} a_j^{i+1} + a_{n+1}^i}$, e.g. $\boxed{a_1 = \sum_1^n b_i + a_{n+1}}$

1. Generalization to Infinite Case

Since $a_n - a_{m+1} = \sum_n^n b_i$,

the generalization is to an infinitessimal terminus a_∞

$$a_n - a_\infty = \sum_n^\infty b_i \text{, or } a_n = \sum_n^\infty b_i + a_\infty$$

Here a_∞ is $\frac{1}{\infty}$, the infinitely small.

2. "Operationalized" or "Constructive" Generalization

$$\forall \epsilon \, \exists \delta \quad a_n = \sum_{n \leq i \leq \delta} b_i + \epsilon$$

Here ϵ may be as small as you like yet not literally an infinitessimal

3. Modern generalization:

$$a_n = \sum_n^\infty b_i$$