

Leibniz: Sum of an Infinite Series

Definition $\{a_j^1\}_1 = \{ \frac{1}{j} \}_1$. Let $\{a_j^1\}$ be a_j

Here $\{a_j^1\} = \{ \frac{1}{j} \} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots$

Definition $\{a_j^{i+1}\} = \{a_j^i - a_{j+1}^i\}$

Here $\{a_j^2\} = \{a_j^{1+1}\} = \{a_j^1 - a_{j+1}^1\}$
 $= \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \dots$

Here $\{a_j^3\} = \{a_j^{2+1}\} = \{a_j^2 - a_{j+1}^2\}$
 $= \frac{1}{3}, \frac{1}{12}, \frac{1}{30}, \frac{1}{60}, \dots$

Let $\{a_j^2\}$ be b_j , $\{a_j^3\}$ be c_j etc.

$$b_n = a_n - a_{n+1}$$

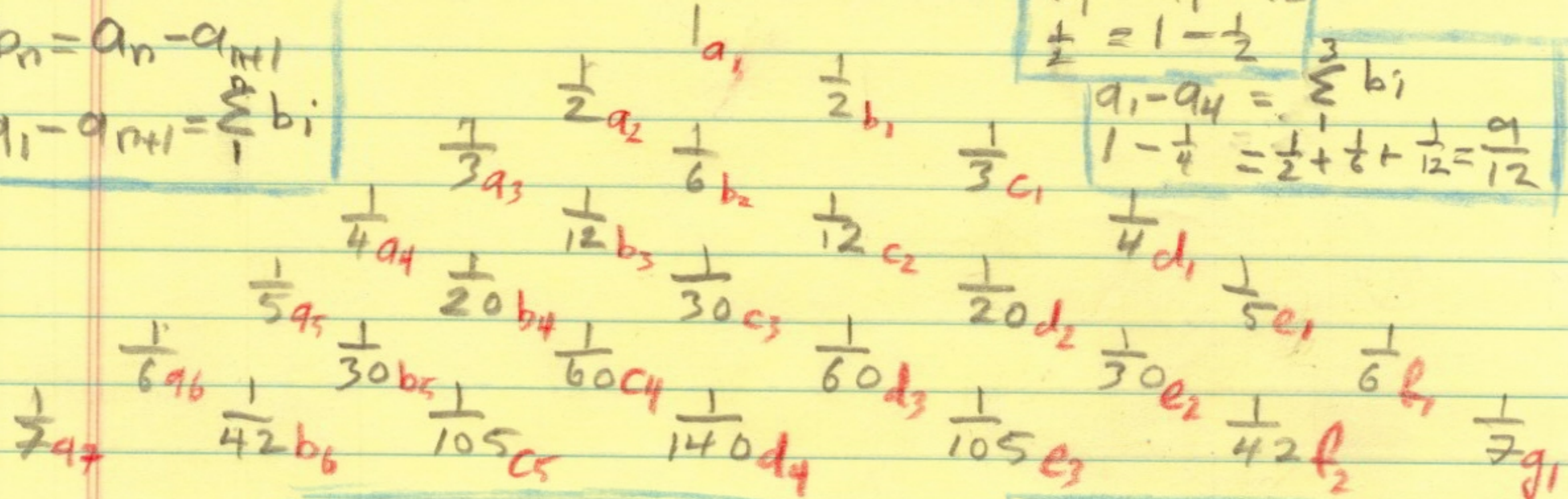
$$a_1 - a_{n+1} = \sum_{i=1}^n b_i$$

$$b_1 = a_1 - a_2$$

$$\frac{1}{2} = 1 - \frac{1}{2} = \sum_{i=1}^1 b_i$$

$$a_1 - a_4 = \sum_{i=1}^3 b_i$$

$$1 - \frac{1}{4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12}$$



Theorem $\sum_{j=1, \dots, n}^{i+1} a_j^i = a_1^i - a_{n+1}^i$ e.g. $\sum_{i=1}^n b_n = a_1 - a_{n+1}$

$$a_1^i = \sum_{j=1, \dots, n}^{i+1} a_j^i + a_{n+1}^i$$

e.g. $a_1 = \sum_{i=1}^n b_i + a_{n+1}$

1. Generalization to Infinite Case

$$\text{Since } a_n - a_{m+1} = \sum_n^n b_i,$$

The generalization is to an infinitesimal term a_∞

$$a_n - a_\infty = \sum_n^\infty b_i, \text{ or } a_n = \sum_n^\infty b_i + a_\infty$$

Here a_∞ is $\frac{1}{\infty}$, the infinitely small.

2. "Operationalized" or "Constructive" Generalization

$$\forall \epsilon \exists \delta \exists N \quad a_n = \sum_{n \leq i \leq \delta} b_i + \epsilon$$

Here ϵ may be as small as you like yet not literally an infinitesimal

3. Modern generalization:

$$a_n = \sum_{n \leq i < \infty} b_i$$